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PART II

DIFFERENTIAL-GEOMETRIC ASPECTS OF DYNAMICS

BY

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DIFFERENTIAL-GEOMETRIC ASPECTS OF DYNAMICS

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INTRODUCTION

The relations between mathematics and physics have been presented so frequently and so adequately in recent years, that further discussion would seem unnecessary. Mathematics, however, is too often taken to be analysis, and the role of geometry is neglected. Geometry may be viewed either as a branch of pure mathematics, or as the simplest of the physical sciences. For our discussion we choose the latter point of view: geometry is the science of actual physical or intuitive space. All physical phenomena take place in space, and hence necessarily present geometric aspects. We confine our discussion to mechanics, and consider the rôle of geometry in mechanics.

The fundamental concepts of mechanics are: space, time, mass, and force. Certain preliminary theories deal with some instead of all these concepts. Space by itself gives rise to pure geometry with all its subdivisions. According to Sir William Rowan Hamilton, algebra is the science of pure time; in fact time is the simplest one-dimensional manifold suggesting the notion of real number, the continuum, the foundation of analysis. Neither mass by itself, nor force by itself, gives rise to an independent theory, for these notions cannot be considered without considering space also.

Space and time together give rise to kinematics. If we do not consider velocities and accelerations, but only displacements (that is, initial and terminal positions without introducing

continuous motion from one to the other), we obtain Ampère's "geometry of motion," which belongs to pure geometry rather than to kinematics.

Space and mass give rise to a separate discipline which may be called the geometry of masses. This deals with centers of gravity, moments of inertia, and moments of higher type, which have been studied extensively in recent years, especially by the Italian mathematicians.

Space and force are the essential concepts employed in rigid statics. Mass and time are not necessary in this theory, which deals essentially with the equivalence and reduction of systems of vectors. The remaining combinations, mass and time, force and time, mass and force, do not produce separate theories, since they can not be discussed without introducing also the concept of space.

Consider then space, time, and mass. The principal development along this line is Hertz's remarkable "geometry and kinematics of material systems," a theory entirely independent of the concept of force.

The other combinations of three of the four concepts have not produced separate developments.

Finally, we have the theory which involves all four concepts simultaneously, namely, kinetics.

Although the geometric aspects of the preliminary theories are very interesting and important, it is not our intention to review the progress which has been made in this line. We mention only Ball's theory of screws, Study's Geometrie der Dynamen, and the law of duality connecting kinematics and statics—a law which is not dynamical, but purely geometric.

The notion of vector is of course fundamental in many of these theories. We recall the fact that there are three distinct types of vector used in mechanics: the *free* vector, the *sliding* vector, the *bound* vector. These three types differ with respect to the definition of equivalence. In the first theory, two vectors are regarded as equivalent when they have the same length

and direction (including sense). Such free vectors are employed in combining translations, or forces acting at a point. A free vector in space has three coordinates. The sum of any number of vectors is a vector.

In the second theory, dealing with sliding vectors, two vectors are equivalent only when they have the same line as well as the same length and sense. Such vectors are used in the statics of rigid bodies. The sliding vector in space has five coordinates. A system of these vectors can not usually be reduced to a single vector. The most general system depends in fact on six essential parameters: it is a new geometric element which may be represented either as a screw or a dyname.

Finally, in the third type of vector theory, two vectors are not equivalent unless they have the same initial point and same terminal point, that is the vector is completely bound. Such a vector in space depends on six coordinates. The most general system depends on twelve essential parameters. This is the theory required in the developments of *astatics*.

Statics and kinematics have given rise to very extensive geometric developments; but kinetics still is thought of almost exclusively as a matter of differential equations. Lagrange, in the famous preface to his Mécanique Analytique, stated that no diagrams would be found in his work: "Lovers of analysis will thank me for adding a new branch to that science." The special object of these lectures will be to point out some of the geometric aspects of kinetics, especially properties of the trajectories described in arbitrary fields of force. While the investigations connected with statics and kinematics are mainly of algebraic-geometric character, our kinetic discussions relate to infinitesimal properties, tangents, distribution of curvature, osculating conics, and so on: we shall deal chiefly with the *differential geometry of systems of trajectories*. It is essential to observe that the properties considered relate not to the individual curves, but to the infinite systems of curves.

To emphasize this point, consider the motion of a particle

in a plane field of force, the force depending only on the position of the point. For given initial conditions, the particle will move on a definite curve; taking all possible initial conditions, we shall obtain a triply infinite system of curves. A single curve obviously has no peculiarities, for a particle may be made to describe any given curve by selecting a proper force, varying from point to point of that curve. The system of curves, however, will have intrinsic peculiarities, for if a triply infinite system of curves is given at random, it will not usually be possible to find any field of force such that every particle moving in that field will describe one of the given curves; there is, for instance, no field of force which produces as its trajectories all the circles of the plane.

The simplest general property of the system of trajectories is as follows: If a particle is started at a given position in a given direction with all possible initial speeds into a field of force, a single infinity of trajectories will be obtained, one for each value of the speed; construct for each of these curves the parabola having four-point contact (osculating parabola); the foci of these parabolas will always lie on a circle passing through the given initial point. An equivalent statement is that the directrices of these parabolas will always be concurrent. In space we employ osculating spheres and find that the locus of the centers is a straight line.

A completely characteristic set of properties, for both the plane and space, is given in Chapter I. It is thus possible to tell when a given system of curves can serve as a system of dynamical trajectories. A method is obtained for constructing the field from its trajectories. If say a handful of particles is thrown into an unknown field (the force acting at any point depending only on the position of the point) and if a photograph of the totality of paths is taken, then, without any record of velocity or any observation of time, the field can be constructed. In particular it is possible, by simple geometric tests, to distinguish conservative from non-conservative fields.

Chapter II deals with the geometry of conservative forces. Here the energy equation allows us to group the trajectories into "natural families." Such a family is obtained most concretely as the totality of ∞^4 rays or paths of light in any medium where the index of refraction varies continuously from point to point. The geometric characterization is first given by two simple properties relating to circles of curvature; and then by a new converse of the theorem of Thomson and Tait. It is seen, for example, that if a candle is placed in the atmosphere or in any gas of variable density, the ∞^2 rays emitted by it, which may be curves of very complicated shape, will necessarily have these properties: (A) the circles of curvature constructed at the given source all meet at a second point; (B) three of these circles have four-point (instead of merely three-point) contact with their curves, and these three are mutually orthogonal; (C) the ∞^2 rays form a normal congruence, that is, admit ∞^1 orthogonal surfaces. Natural families are characterized either by (A) and (B), or by (A) and (C).

These results are applied to the propagation of waves in any isotropic medium. A second and more complicated converse question suggested by the Thomson-Tait theorem is discussed. Some interesting optical theorems are given a geometric formulation, but the converse problems are left unsettled. The final section deals with the "general problem of dynamics" in the sense of the French writers.

The third chapter deals with transformation theories. It is interesting to notice how the most important groups of geometry, the projective and the conformal, play essential rôles in dynamics, the former in connection with arbitrary fields, the latter in connection with conservative fields and natural families. The infinitesimal contact transformations of mechanics, and a new group of space-time transformations are also discussed.

The chief subject of Chapter IV is a simple problem in constrained motion, which includes, and hence serves to unify, the theories of trajectories, brachistochrones, catenaries, and velocity

curves in an arbitrary field of force. Complete characterizations are given. Curves of constant pressure and tautochrones are treated only briefly.

Chapter V includes brief discussions of more complicated problems in motion, for example, the effect of a resisting medium on the geometric character of the system of trajectories; the motions of any number of interacting particles (the results being of course applicable to the problem of three bodies); finally, forces depending not only on position but also on the time, both trajectories and space-time curves being studied. The latter are constructed, in the sense of Minkowski, in the four-dimensional space (x, y, z, t), but the application made is to ordinary dynamics, not to electrodynamics or relativity theory.

The main results of the first two chapters (in particular the complete characterizations of general systems of trajectories and of natural families) were first given by the writer in a series of four papers published in the *Transactions* of this Society (1906–1910). Some of the other results are given in notes published in the *Bulletin*. The last two chapters, as well as many sections of the other chapters, deal with hitherto unpublished results.

CHAPTER I

TRAJECTORIES IN AN ARBITRARY FIELD OF FORCE

§§ 1–8. Trajectories in the Plane

1. We consider first the motion of a particle in the plane under the action of any positional field of force. The general equations of motion are

$$m\frac{d^2x}{dt^2} = \varphi(x, y), \quad m\frac{d^2y}{dt^2} = \psi(x, y),$$

where m is the mass and φ, ψ are the rectangular components of the force acting at any point x, y. There is no loss of generality in assuming the mass of the particle to be unity, so we write*

$$\ddot{x} = \varphi(x, y), \qquad \ddot{y} = \psi(x, y). \tag{1}$$

The particle may be started from any position (x_0, y_0) with any velocity $(\dot{x}_0, \dot{y}_0)$. A definite trajectory is then described. Since the same curve may be obtained by starting from any one of its ∞^1 points, the total number of trajectories, for all initial conditions, is ∞^3. The differential equation of the third order representing this system of trajectories, found by eliminating the time from (1), is

$$(\psi - y'\varphi)y''' = \{\psi_x + (\psi_y - \varphi_x)y' - \varphi_y y'^2\}y'' - 3\varphi y''^2. \tag{2}$$

This is not an arbitrary differential equation of the third order. Hence the system of trajectories generated by an arbitrary field of force must have peculiar geometric properties, which translate the peculiar analytic form of (2).

* *The following notation is employed throughout these lectures:* Dots indicate total differentiation with respect to the time t; primes indicate total differentiation with respect to x; subscripts x and y indicate partial differentiation; finally, the subscript s indicates total differentiation with respect to the arc length s.

2. Before stating these, we remark that a more intrinsic basis for the discussion is obtained by decomposing the acting force into components normal and tangential to the path, instead of parallel to x and y axis as in (1). Denoting these components by N and T respectively, the equations of motion are

$$v^2/r = N, \quad vv_s = T, \tag{3}$$

where v denotes the speed, s the arc length, and r the radius of curvature. By differentiating the first of these equations with respect to s, and comparing with the second equation, we can eliminate v, obtaining

$$(rN)_s = 2T, \tag{4}$$

a relation which defines the trajectories and is equivalent to (2).

To reduce this to a more explicit form, we introduce an auxiliary vector, completely determined by the given field of force, namely the space derivative of the force (considered of course as a vector). The normal and tangential components of the force vector are

$$N = \frac{\psi - y'\varphi}{\sqrt{1 + y'^2}}, \quad T = \frac{\varphi + y'\psi}{\sqrt{1 + y'^2}}; \tag{5}$$

the corresponding components of the new vector are

$$\begin{aligned} \mathfrak{N} &= \frac{\psi_s - y'\varphi_s}{\sqrt{1 + y'^2}} = \frac{\psi_x + (\psi_y - \varphi_x)y' - \varphi_y y'^2}{1 + y'^2}, \\ \mathfrak{T} &= \frac{\varphi_s + y'\psi_s}{\sqrt{1 + y'^2}} = \frac{\varphi_x + (\varphi_y + \psi_x)y' + \psi_y y'^2}{1 + y'^2}. \end{aligned} \tag{6}$$

While the new vector is the s derivative of the force vector, its components are obviously not the same as the s derivatives of the old components: the correct relations are found to be

$$N_s = \mathfrak{N} - \frac{T}{r}, \quad T_s = \mathfrak{T} + \frac{N}{r}. \tag{7}$$

These formulas are sufficient for the discussion of trajectories.* By means of (7) we can reduce (4) to the form

$$(8) \qquad Nr_s = -\mathfrak{N}r + 3T.$$

This is the fundamental intrinsic representation of the system of ∞^3 trajectories connected with a given field of force.

From it we may derive very simply a number of geometric properties. But in dealing with the converse questions which arise, and in proving the completeness of the set obtained, it is more convenient to use the equivalent cartesian representation, that is, equation (2).†

3. *The Five Characteristic Properties in the Plane.*—The system of trajectories generated by any positional field of force in the plane has the following set of properties, and conversely, any system of ∞^3 curves which has these properties will be a system of dynamical trajectories.

* In some of the later discussions we shall need also the space derivatives of $\mathfrak{N}$ and $\mathfrak{T}$, which may be written in the form

$$\mathfrak{N}_s = \mathfrak{N}_1 + \frac{\mathfrak{N}_2}{r}, \qquad \mathfrak{T}_s = \mathfrak{T}_1 + \frac{\mathfrak{T}_2}{r},$$

where

$$\mathfrak{N}_1 = \frac{\psi_{xx} + (2\psi_{xy} - \phi_{xx})y' + (\psi_{yy} - 2\phi_{xy})y'^2 - \phi_{yy}y'^3}{(1 + y'^2)^{3/2}},$$

$$\mathfrak{N}_2 = \frac{\psi_y - \phi_x - 2(\phi_y + \psi_x)y' + (\phi_x - \psi_y)y'^2}{1 + y'^2},$$

$$\mathfrak{T}_1 = \frac{\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3}{(1 + y'^2)^{3/2}},$$

$$\mathfrak{T}_2 = \frac{\phi_y + \psi_x + 2(\psi_y - \phi_x)y' - (\phi_y + \psi_x)y'^2}{1 + y'^2}.$$

The functions ϕ, ψ depend only on the position of the particle; the auxiliary intrinsic functions N, T, $\mathfrak{N}$, $\mathfrak{T}$, $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{T}_1$, $\mathfrak{T}_2$, defined above, depend also upon the direction of motion; finally, N_s, T_s, $\mathfrak{N}_s$, $\mathfrak{T}_s$ depend upon the curvature of the path. Cf. *Bull. Amer. Math. Soc.*, vol. 15 (1909), p. 475.

† Cf. *Trans. Amer. Math. Soc.*, vol. 7 (1906), pp. 401–424. The result contained in property IV of § 3 gives this simple, but apparently overlooked, dynamical theorem: If a particle starts from rest, the initial curvature of the path described is one third of the curvature of the line of force through the initial position.

I. If for each of the ∞^1 trajectories passing through a given point in a given direction we construct the osculating parabola, at the given point, the locus of the foci of these parabolas is a circle passing through that point.

II. The circle that corresponds, according to property I, to a lineal element, is so situated that the element bisects the angle between the tangent to the circle and a certain direction fixed for the given point (the direction of the force acting at the given point).

III. In each direction at a given point there is one trajectory which has four-point contact with its circle of curvature: the locus of the centers of the ∞^1 hyperosculating circles constructed at the given point is a conic passing through that point in the fixed direction described in property II.

IV. With any point O there is associated a certain conic passing through it as described in property III. The normal to the conic at O cuts the conic again at a distance equal to three times the radius of curvature of the line of force passing through O. (The lines of force are defined geometrically by the fact that the tangent at any point has the direction associated with that point in accordance with property II.)

V. When the point O is moved, the associated conic referred to above changes in the following manner. Take any two fixed perpendicular directions for the x direction and the y direction; through O draw lines in these directions meeting the conic again at A and B respectively. Also construct the normal at O meeting the conic again at N. At A draw a line in the y direction meeting this normal in some point A', and at B draw a line in the x direction meeting the normal in some point B'. The variation property referred to takes the form

$$\frac{\partial}{\partial x}\frac{1}{AA'}+\frac{\partial}{\partial y}\frac{1}{BB'}+\frac{\omega\omega_{xy}-\omega_x\omega_y}{3\omega^2}=0,$$

where AA' and BB' denote distances between points, and where ω denotes the slope of the lines of force relative to the chosen

x direction. This is true for any pair of orthogonal directions,

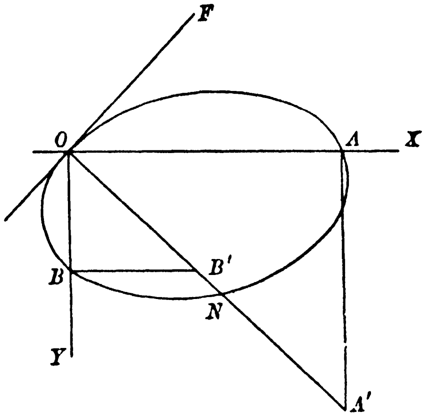

FIG. 1.

and therefore really expresses an intrinsic property of the system of curves.

4. The most general system of ∞^3 curves in the plane is represented by an arbitrary differential equation of the third order

$$(F_0) \qquad y''' = f(x, y, y', y'').$$

It thus involves *one* arbitrary function of *four* arguments.

A system of dynamical trajectories, on the other hand, is represented by an equation of the particular form

$$(F_V) \quad (\psi - y'\varphi)y''' = \{\psi_x + (\psi_y - \varphi_x)y' - \varphi_y y'^2\}y'' - 3\varphi y''^2,$$

and thus involves *two* arbitrary functions of *two* arguments. These are the only systems having all five properties I–V.

It is interesting to notice just how the successive imposition of the properties gradually narrows down the general form (F_0) to the particular form (F_V).

5. *The most general system having property I is found to be*

$$(F) \qquad y''' = G(x, y, y')y'' + H(x, y, y')y''^2.$$

It thus involves *two* arbitrary functions of *three* arguments. This type of course includes the dynamical type as a very special case. It arises in a number of different geometric and physical investigations. It has therefore its own interest. The characteristic property may be stated in various ways, all of course equivalent to the original form: (I) The osculating parabolas of the trajectories passing through a given point in a given direction have the foci situated on a circle passing through the given point. Five equivalent forms are as follows:

I (2). The directrices of the osculating parabolas form a pencil It follows that there exists a point (the vertex of this pencil) from which all the parabolas subtend an angle of 90°.

I (3). If for each of the trajectories considered, we construct the center of curvature of its evolute, the locus of the centers thus obtained is a parabola passing through O, and having its axis parallel to the given initial direction.

I (4). For each of the trajectories, construct the osculating equiangular spiral. The locus of the centers of the poles of these spirals is a circle passing through O.

I (5). Construct for each of the trajectories the axis of deviation, that is the line bisecting the chords of the curve which are parallel and infinitesimally close to the tangent. The tangent of the angle between the varying axis of deviation and the fixed normal is a linear function of the radius of curvature.

I (6). The derivative of the radius of curvature with respect to the arc length is a linear integral function of the radius of curvature. This is practically a restatement of (5), since for any curve the derivative of the radius of curvature is known to be equal to three times the tangent of the angle of deviation. But in this form it has the advantage of being valid, not only in the plane, but in space of three and in fact any number of dimensions.

If in addition to property I, we impose property II, the function $H(x, y, y')$ is specialized to

$$H = \frac{3}{y' - \omega(x, y)},$$

Thus the most general system with properties I and II is

$$(F_{\text{II}}) \qquad (y' - \omega)y''' = (y' - \omega)Gy'' + 3y''^2$$

where G is any function of x, y, y', and ω is any function of x, y. The type thus involves *one* arbitrary function of *three* arguments and one arbitrary function of two arguments.

6. *Systems with Properties I, II, III.*—Imposing also property III, we find that $G(x, y, y')$ must be of the special form

$$G = \frac{\lambda y'^2 + \mu y' + \nu}{y' - \omega}.$$

Thus the most general system of ∞^3 curves having properties I, II, III is represented by

$$(F_{\text{III}}) \qquad (y' - \omega)y''' = (\lambda y'^2 + \mu y' + \nu)y'' + 3y''^2,$$

involving *four* arbitrary functions ω, λ, μ, ν each of the *two* arguments x, y.

This type may be characterized by the following properties which are then equivalent to I, II, III.

I (2). For a given lineal element, the directrices of the ∞^1 osculating parabolas pass through a common point D.

II (2). When the lineal element turns about the given point O, the point D describes a straight line passing through O.*

III (2). The correspondence between the range of points D and the pencil of elements through O is one-to-two of the special form

$$\frac{3}{2d} = \lambda \sin^2 \theta + \mu \sin \theta \cos \theta + \nu \cos^2 \theta,$$

where d denotes the distance OD, and θ is the angle between the element and fixed direction of OD.

* In the dynamical case this line OD is perpendicular to the force vector acting at O. For certain special fields the point D may remain fixed: this happens only when the components of the force are conjugate harmonic functions, that is when the field is of the type termed "analytic" by Lecornu.

7. If now we add the properties IV and V, the four functions ω, λ, μ, ν appearing in (F_{III}) must obey the relations

$$(F_{\text{IV}}) \qquad \lambda\omega^2 + \mu\omega + \nu + \omega_x + \omega\omega_y = 0,$$
$$(F_{\text{V}}) \qquad (\omega_y + \lambda\omega + \mu)_y - \lambda_x = 0.$$

Thus the general system having properties I–IV involves *three* arbitrary functions of x, y; while that having all five properties involves *two* such functions.

By integrating these relations, we may express the four functions in terms of two arbitrary functions φ, ψ as follows:

$$\omega = \frac{\psi}{\varphi}, \qquad \lambda = \frac{\varphi_y}{\varphi}, \qquad \mu = \frac{\varphi_x - \psi_y}{\varphi}, \qquad \nu = -\frac{\psi_x}{\varphi}.$$

These values, substituted in the type (F_{III}), actually give rise to the type

$$(\psi - y'\varphi)y''' = \{\psi_x + (\psi_y - \varphi_y)y' - \varphi_y y'^2\}y'' - 3\varphi y''^2,$$

and thus the proof is completed that the set of five properties characterizes the dynamical type.

In connection with the statements I (2), II (2), III (2), property IV may be formulated as follows:

IV (2). In the correspondence described in III (2), if the element approaches the direction of the force the corresponding distance OD has for its limiting value 3/2 the radius of curvature of the line of force passing through O. It is to be remembered that the direction of the force, and hence also the lines of force, are defined purely geometrically in terms of the given triply infinite system of curves by the fact that at any point O in the plane the "direction of the force" is perpendicular to the line described as the locus of D in the above equivalent II (2) of property II.

In the same line of ideas it would be possible to find an equivalent for property V (thus completing the characterization), but the result V (2) cannot be put into simple form. The original form V may be criticized as inelegant because in it reference is made to a system of cartesian axes. Of course the result

expresses an intrinsic property since it is true for all systems of axes. It would certainly be desirable to restate the result in intrinsic language. This can be done, for instance, by introducing the distances cut off by the conic described in IV, not only on the normal ON, but also on the two lines inclined at an angle of 45° to that normal. However it does not seem possible to obtain a statement which is both simple and intrinsic in form.

8. Of course many other properties of trajectories may be obtained, either by reasoning synthetically from the five fundamental properties, or by reasoning analytically from the fundamental differential equation. We state only a few samples.

If we shoot particles from a given position in a given direction with variable speed, the center of curvature of the resulting trajectories describes a straight line (the normal) and the focus of the osculating parabola simultaneously describes a circle (by property II), in such a way that the two ranges (one linear, the other circular) are homographically related; furthermore the given point, which is on both ranges, corresponds to itself.

If we shoot from the same position in a direction perpendicular to that previously employed, the new focal circle will be tangent (at the given point) to the former focal circle. Conversely if two focal circles, for the same point, are tangent, the initial directions to which they correspond will be perpendicular to each other.

We shall make use of the following properties which describe the disposition of the ∞^1 focal circles constructed at a given point. The two results which follow are geometrically equivalent, and either may be substituted for property III in the fundamental set.

III (3). If for each of the elements at a given point we construct the corresponding focal circle, the locus of the centers of the ∞^1 circles thus obtained is a conic with one focus at that point.

III (4). The envelope of the ∞^1 focal circles is always a circle.*

* This enveloping circle is in general position: it does not usually have its center at the given point. This simple position arises only when the force is of the Lecornu type.

§ 9. Actual and Virtual Trajectories

9. If we consider the motion of a cannon ball in a given vertical plane under the action of gravity assumed constant, the triply infinite system of trajectories consists of parabolas with vertical axes. We do not, however, obtain all vertical parabolas, represented by the differential equation of the system of trajectories, which is here $y''' = 0$, but only those whose concavity is directed downwards. The other vertical parabolas, with concavity directed upwards, satisfy the same differential equation, and it is therefore convenient to include them in the system studied. We thus have a distinction of *actual* and *virtual* trajectories. The latter are the actual trajectories corresponding to gravity reversed in direction.

In an arbitrary field of force the same distinction arises. The complete system of trajectories is composed of the actual trajectories corresponding to the given force, and the virtual trajectories which are the actual trajectories corresponding to the reversed field. It is obvious that the system of trajectories is not changed if the force acting at every point is multiplied by a constant. If we were considering only actual trajectories, it would be necessary to restrict this constant to positive values, but as we include both actual and virtual, the constant factor may also be negative. (Of course the constant must not be zero, since then the force would vanish and we should obtain the trivial system of straight lines.)

It is easy to show that the virtual trajectories corresponding to the given field may be found by giving the initial speed of the particle a pure imaginary value. The cannon ball could be made to describe a parabola with its concavity directed upwards if only some kind of powder could be invented which would cause its initial speed to be imaginary!

In discussing the general geometric properties of trajectories, we had in mind of course the complete system as defined by the differential equation. Consider for example property I: for any

given lineal element the locus of the foci of the parabolas osculating the corresponding trajectories is a circle through the given point. The question arises, what part of this circle corresponds to the actual trajectories. It is easily found to be the arc of the circle cut off by the initial direction line (the common tangent of the trajectories considered) on that side which is indicated by the force vector. Thus, if we confined our discussion to actual trajectories, the focal locus would be, not a *circle,* but an *arc of a circle,* the arc running from the given point O to a certain terminal point A. If we consider all elements through O the locus of the corresponding terminal point A is found to be a conic passing through O in the direction of the force vector.*

For a given element, the point A, which separates the actual from the virtual, may be defined as the limiting position of the focus of the osculating parabola as the initial speed becomes infinitely large. The osculating parabola in this limiting case becomes a straight line, but the focus has a definite limiting position.

An analogous distinction, into actual and virtual, presents itself also in the theories of brachistochrones, catenaries, and tautochrones. The differential equations of the systems of curves are satisfied by both the actual and virtual curves, and it is the complete systems that we refer to in all our discussions unless the contrary is explicitly mentioned.

§§ 10–15. Trajectories in Space

10. Consider the motion of a particle, which we may take to be of unit mass, in an arbitrary positional field of force. The equations of motion are

$$(1) \qquad \ddot{x} = \varphi(x, y, z), \qquad \ddot{y} = \psi(x, y, z), \qquad \ddot{z} = \chi(x, y, z).$$

The particle may be started from any position, in any direction, with any speed: its motion is then determined by the field of

* This conic is not the same as the conic arising in property III.

force, and it describes a definite trajectory. The totality of trajectories constitutes a definite system of ∞^5 curves. (We exclude the case where the force vanishes at every point, the trajectories then being merely the ∞^4 straight lines.)

What are the properties of such quintuply infinite systems of curves? Obviously an arbitrary system of space curves cannot be obtained as the totality of trajectories connected with any field of force. In fact the most general system of ∞^5 curves (assuming that ∞^1 curves pass through any point of space in any direction) would be represented by a pair of differential equations, one of the third order and one of the second order, of the general form

$$(2) \quad y''' = f(x, y, z, y', z', y''), \qquad z'' = g\,(x, y, z, y', z', y''),$$

thus involving two arbitrary functions of *six* arguments; while the dynamical type involves merely three arbitrary functions of *three* arguments. The differential equations representing the dynamical type, obtained by eliminating the time from the equations of motion, may be written in the form

$$(3) \quad \begin{aligned} (\psi - y'\varphi)y''' &= \begin{vmatrix} 1 & \varphi_x + y'\varphi_y + z'\varphi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix} y'' - 3\varphi y''^2, \\ (\psi - y'\varphi)z'' &= (\chi - z'\varphi)y''. \end{aligned}$$

The question is to express the peculiar form of these equations in simple geometric language.

The interpretation of the second equation is obvious: the osculating plane of the path passes not only through the given initial direction $1 : y' : z'$, but also through the fixed direction $\varphi : \psi : \chi$; that is, the osculating plane always passes through the direction of the force acting at the given point. The other properties are not obvious;* they take into account the form of the differential equation of third order.

* The simplest of these, property II below and certain consequences, were first stated in the author's note published in the *Bull. Amer. Math. Soc.*,

We cannot now, as in the case of the plane discussion, employ osculating parabolas, since our curves are twisted. Three consecutive points of a curve determine an osculating circle. What do four consecutive points determine? No simple type of osculating curve is available, so we shall make use of the osculating sphere. The results are therefore quite different in form from those obtained in the two-dimensional theory.

11. *The Four Properties in Space.*—In order that a system of ∞^5 space curves, of which ∞^1 pass through each point in each direction, shall be identifiable with the system of trajectories generated by a positional field of force, it is necessary and sufficient that it shall have the following four purely geometric properties:

I. The osculating planes of the ∞^3 curves passing through a given point form a pencil; that is, all the planes pass through a fixed direction.

II. The osculating spheres of the ∞^1 curves passing through a given point in a given direction form a pencil; their centers thus lie on a straight line.

III. The straight lines which correspond, in accordance with II, to all the ∞^2 directions at a given point, form a congruence (of order one and of class three) consisting of the secants of a twisted cubic curve; which cubic furthermore passes through the given point in the direction fixed by property 1.

IV. The associated plane systems S', determined by the given space system in the manner described below, have the five geometric properties characteristic of a system of plane dynamical trajectories. Consider the given system of ∞^5 space curves in connection with any plane p. Through any point of p there pass ∞^2 curves of the given system which are tangent to the plane. Project the differential elements of the third order belonging to these space curves orthogonally upon p, thus obtaining ∞^2

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vol. 12 (1905), pp. 71–74. Somewhat simplified proofs were then given by Cesàro, in a paper published shortly before his death, in the *Memorie di Torino* (1905). The complete set of properties appeared in the *Trans. Amer. Math. Soc.*, vol. 8 (1907), pp. 121–140.

plane differential elements of the third order at the selected point. Applying this process to all points of p, we have a defined set of ∞^4 differential elements of the third order. These elements define a certain differential equation of the third order, and thus determine a system of ∞^3 integral curves. This we term the associated system in the plane p. The space system has the property that every one of these plane systems associated with it is a system of dynamical trajectories, and therefore has the five properties stated in § 3, which we here denote by I_p–V_p in order to avoid confusion with the four spatial properties.

These four properties are ordinally independent: no one can be derived from those which precede it. The question of absolute independence is left open: it is quite probable that IV is sufficiently strong to furnish a complete characterization by itself.

12. The most general system having property I involves one arbitrary function of six arguments besides two functions of three arguments. These systems have the following properties, which are of course consequences of property I.

The ∞^1 curves passing through a given point in a given direction have not only the same osculating plane, but also the same *torsion*.

If the torsion is given the corresponding initial directions form a quadric cone. In particular such a cone defines the directions of those curves, through the given point, which admit hyperosculating planes.

If for each of these curves we construct, at the common point, the *related helix** (that is the helix which agrees with the curve in osculating plane, curvature, and torsion), the axes of the helices so obtained generate a cylindroid.

13. The most general system with properties I and II involves two arbitrary functions of five arguments, besides two functions of three arguments. Two further statements, each equivalent to II, are as follows:

*An osculating helix, that is, one having four-point contact with the curve, does not in general exist.

If for each of the ∞^1 curves defined by a given lineal element we construct the osculating circle and the osculating sphere, the distance between the center of the circle and the center of the sphere varies as a linear integral function of the radius of curvature.

For the same set of ∞^1 curves, the derivative of the radius of curvature with respect to the arc length can be expressed as a linear integral function of the radius of curvature.

This last form has the advantage of being valid in space of two or any number of dimensions. On this basis, however, it would be difficult to formulate equivalents for the higher properties, so as to obtain a complete characterization.

14. Property III is perhaps the most interesting result obtained. The most general system having this property in addition to I and II is represented by a pair of differential equations involving ten arbitrary functions of three arguments.

One may ask what is the significance of the cubic curve (we denote it by Γ), which arises in connection with III. To each point O of space there is related a certain cubic Γ. If we shoot from O in every direction with every speed, we obtain ∞^3 trajectories. Each of these has an osculating sphere with a definite center C. To each of the trajectories there corresponds one center C. Usually the center C determines the trajectory. However if C lies on the curve Γ, there are ∞^1, instead of one, corresponding trajectory: in this case in fact the initial direction may be any direction perpendicular to the line joining O and C.* Thus the curve Γ may be defined as the locus of those points which may serve as centers for more than one trajectory through the given point O.

A simple consequence of III is that the locus of the centers of the osculating spheres of the ∞^2 trajectories touching a given plane at a given point is a quadric surface.

* Two trajectories through O have the same osculating sphere only if the initial speed is the same, and if the line through O perpendicular to the initial elements meets the cubic Γ.

If the plane varies, the given point being held fixed, the ∞^2 quadrics obtained form a linear system.*

The properties so far considered relate to the curves through a given point O. If we have ∞^3 curves passing through a point O, ∞^1 in each direction, and if, at that point, properties I, II, III are fulfilled, it will not usually be possible to generate the curves as trajectories in any field of force. All that follows is that the relations between y', z', y'', z'', y''', z''' are of precisely the same form as those holding for trajectories; and therefore it is possible to find (in infinitely many ways) a field of force such that each of the ∞^3 trajectories passing through the given point shall have contact of the third order with some one of the given curves.

15. In order to cause our system to be of the dynamical type, it is necessary to restrict the ten arbitrary functions involved in the type characterized by I, II, III so that only three arbitraries remain, namely, the components φ, ψ, χ defining the field of force. This is the rôle of property IV, which states that in any plane p the associated system S is of the plane dynamical type. An equivalent statement is as follows:

IV (2). If the ∞^2 space curves touching any plane p at any point O are projected orthogonally upon p, the plane curves thus obtained possess the properties I_p, II_p, III_p; when the point O varies in p, the direction associated with it by II_p, and the conic associated with it by III_p, vary in accordance with the restrictions expressed in IV_p and V_p.

It may be remarked that the first half of this statement holds for all space systems having properties I, II, III; in fact all such systems have also property IV_p. The real restriction is in V_p. It is also sufficient to consider, instead of all planes p, merely those of a triply orthogonal set.

§§ 16–25. The Inverse Problem of Dynamics: A Method of Geometric Exploration

16. The usual direct and inverse problems arising in dynamics are: first, given the force acting on a particle, to find its motion;

* On the other hand if we vary the given point, keeping the plane fixed, no simple result is obtained: the ∞^2 quadrics constitute an arbitrary family.

and second, given the motion of a particle, to find the force acting on it. The first problem is solved by integrating the differential equations of motion. The second is solved by differentiating the coordinates of the point with respect to the time.

Suppose, however, that we are given only the path described by the particle but have no information about the motion along the curve. If merely a single curve is given, the problem of finding the acting force would of course be indeterminate. But if all the trajectories, described by starting particles in a field of force from all initial positions in all directions with all speeds, are given, then the field of force is essentially determined (that is, up to a constant factor). *Hence if we were given a photograph of the entire system of curves generated by some (positional) field of force, without any record of motion or time, it ought to be possible to find the law of the field of force.* This is easily seen to be true analytically; but we wish also a purely geometric solution which will enable us to pass from the given curves to the vector representing the force at each point of the plane (taking first the two-dimensional case). The result gives what may be described as a method for the *geometric exploration of a field of force.*

17. First consider two trajectories passing through the same point O in the same direction. Construct the two osculating parabolas. The circle passing through the point O and the foci of these parabolas will, according to property I, be the focal circle corresponding to the given point and the given direction. Then, according to property II, the direction of the force acting at O will be symmetric to the tangent to this circle at O with respect to the common tangent of the two curves. An equivalent of this construction is to join O to the intersection of the directrices of the osculating parabolas: this line is perpendicular to the direction of the force acting at O.

If we are given two trajectories passing through O in different directions, then the direction of the force at O is not determined. The same is true if we are given three curves with distinct tangents.

18. *If, however, we are given four trajectories with distinct tangents, the force direction is (in general) uniquely determined.*

Consider an arbitrary direction at O, and let us see if it can be the direction of the force acting at that point. Take the image of this direction in the tangent to the first of the given curves; then pass a circle through O in the direction so obtained and through the focus of the corresponding osculating parabola. Doing this for each of the four curves, we obtain four focal circles. *If there exists a circle touching these four, the direction tested is correct.* This follows from property III (4) of § 8. We have then a purely geometric problem: to find a direction at O such that the four circles constructed by means of it shall admit a common tangent circle. We may simplify this problem by inverting the configuration considered with respect to O. We then have, instead of the four focal circles, four straight lines which are to be concyclic. As we change the direction tested, these rotate simultaneously through equal angles about four fixed points, namely, those obtained by inverting the four foci.

Take an arbitrary oriented direction for trial; construct for each of the four inverse foci, a direction parallel to the image of the tangent to the focal circle with respect to the tangent to the corresponding trajectory. We thus obtain four oriented lineal elements, one at each of the inverse foci. The problem is then to rotate these through the same angle α, so that the new elements shall have concyclic lines.* In this position the image of the direction of any one of the four elements in the corresponding tangent at O will give the required direction of the force. The only ambiguity, in general, will be in the sense (arrowhead) of the force: this, however, may be determined separately for actual† trajectories by considerations of concavity and convexity.

19. The direct analytical treatment is as follows. The differential equation of the ∞^3 trajectories of any positional field

* A simple ruler and compass solution of this problem was suggested to the author by Professor Wedderburn.

† See § 9.

of force is of the form

$$(y' - \omega)y''' = (\lambda y'^2 + \mu y' + \nu)y'' + 3y''^2,$$

where λ, μ, ν, ω are functions of x, y (and have therefore fixed numerical values so far as we deal with the ∞^2 curves passing through a given point O), the latter quantity ω representing the slope of the acting force. Each of the four given curves C_1, C_2, C_3, C_4 through the point O determines certain values of the derivatives y', y'', y'''; that is we are given the differential elements of third order

$$y_i', \quad y_i'', \quad y_i''' \qquad (i = 1, 2, 3, 4).$$

Substituting these values we have four linear equations

$$(y_i' - \omega)y_i''' = (\lambda y_i'^2 + \mu y_i' + \nu)y_i'' + 3y_i''^2 \quad (i = 1, 2, 3, 4),$$

from which we can find the values of λ, μ, ν, ω at the given point. *The required direction of the acting force is determined by the slope*

$$\omega = \frac{\begin{vmatrix} 1 & y_i' & y_i'^2 & \dfrac{y_i'y_i''' - 3y_i''^2}{y_i''} \end{vmatrix}}{\begin{vmatrix} 1 & y_i' & y_i'^2 & \dfrac{y_i'''}{y_i''} \end{vmatrix}},$$

where numerator and denominator are determinants of the fourth order.

20. By any of these methods we may determine the direction* of the vector representing the force acting at any point O of the plane. How shall we determine the magnitude of the vector? The determination cannot be absolute, since, as already remarked, two fields that differ by a constant factor have identical trajectories. The magnitude of the vector at any one point may be taken at random, and then the field is completely determined.

This depends upon the simple fact that if we know the *path*

* Of course if *all* the trajectories were given, the direction of the force would be determined immediately by the fact that the curves in that direction have zero curvature.

of a particle and also the *direction* of the force acting at each of its points, then assuming the *magnitude* arbitrarily at one point, it is completely determined at all points. This is an integration problem. We know the force vector at the initial point O, and may decompose it into components N and T, normal and tangent to the given curve. Assuming the mass to be unity, the initial speed is given by

$$v_0^2 = rN,$$

where r is the known radius of curvature. Then from

$$vv_s = T,$$

we may find v_s, the rate of variation of the speed for unit of arc. The speed at any point P of the curve is thus found in the form

$$v = v_0 e^{\int \frac{T}{rN} ds}$$

where all the quantities in the right-hand member are geometrically given. (The integrals throughout are calculated from point O to point P.) If we denote by θ the inclination of the force to the curve, so that $\tan\theta = N/T$, the speed is

$$v = v_0 e^{\int \frac{\cot\theta}{r} ds}.$$

Since the speed, that is the motion, is now known, the magnitude of the force is also known. The components at any point P are

$$N = N_0 e^{\int \frac{2\cot\theta - r_s}{r} ds}, \quad T = N\cot\theta.$$

21. We see that the construction of the field may be carried out without knowing all the ∞^3 trajectories. So far as the direction of the force is concerned, it is sufficient to know at each point of the plane either two trajectories with a common tangent, or four trajectories with distinct tangents. So far as

the magnitude is concerned it is then sufficient to know ∞^1 trajectories through one point O, one for each direction, since we can then integrate from this point to any point of the plane* along some one of the curves.

A field of force is in general determined, and may be constructed, if we know $4\infty^1$ *out of the totality of* ∞^3 *trajectories*, each of the four systems of ∞^1 curves covering the plane (or the region considered) simply, that is, one passing through each point of the plane.

The complete system of ∞^3 trajectories is thus determined in general by four systems of ∞^1 trajectories. Further reduction is possible. In general $3\infty^1$ curves determine the totality, but no simple constructions are then available. If two simply infinite systems of curves (that is, a net of curves) are assigned arbitrarily, a corresponding complete system can be found in a large infinitude of ways: the corresponding field of force is not determined up to a constant, but involves arbitrary functions.

The first and most interesting example of the geometric exploration of a field of force arose in Bertrand's discussion of Kepler's laws. The first of these laws (every planet describes an ellipse having the sun for a focus) is geometric, while the second and third are kinematic (involving the areal velocity and the period). The first law determines all the trajectories, and therefore determines the field of force.† Hence the newtonian law of gravitation can be deduced from the first law alone, instead of as usual, from all three. Bertrand thus concludes that the other two laws are consequences of the first. If Kepler had been a mathematician of the twentieth century, he would have stopped his laborious observational inductions after noting his first law and deduced the other two analytically.

The first law, in Bertrand's discussion, is of course to be taken ideally: not only the actual planets describe conics with a focus at the sun, but every particle starting from any position with

* That is, in some region of the plane—in some neighborhood of O.

† It is assumed, of course, that the force depends only upon the position

any velocity describes such a conic. From what has been stated above it is sufficient to limit the observations to four simply infinite systems of conics in "general" position.

On account of the last phrase, it is easily possible to commit errors in the application of the result. It would be possible to give $4\infty^1$ or even ∞^2 conics in certain special ways, so that the field is *not* determined. (See § 23.)

22. This raises the general question: *How many trajectories may be common to two distinct fields of force?*

The first field, defined by its components φ, ψ, has a system of ∞^3 trajectories with a differential equation

$$(y' - \omega)y''' = (\lambda y'^2 + \mu y' + \nu)y'' + 3y''^2;$$

the second field, with components φ_1, ψ_1, has a system of trajectories given by an analogous equation

$$(y' - \omega_1)y''' = (\lambda_1 y'^2 + \mu_1 y' + \nu_1)y'' + 3y''^2.$$

If there are any solutions in common,* they must satisfy the equation of second order

$$3(\omega - \omega_1)y'' = (y' - \omega)(\lambda_1 y'^2 + \mu_1 y' + \nu_1) - (y' - \omega_1)(\lambda y'^2 + \mu y' + \nu).$$

Two systems of trajectories cannot have more than ∞^2 curves (one through each point in each direction) in common without coinciding. If they have ∞^2 curves in common the differential equation of the second order defining these curves must be of the cubic form†

$$y'' = Ay'^3 + By'^2 + Cy' + D,$$

where the coefficients are functions of x, y.

Usually the solutions of the equation of the second order will

* In addition to straight lines, $y'' = 0$, which are common to *all* systems.

† This form is characterised by the fact that the locus of the centers of curvature of the curves passing through a given point is a special type of cubic curve. Cf. *Amer. Jour. Math.*, 1908, p. 207.

not satisfy either equation of the third order and the two systems will have no curves in common. An example showing that the two systems may actually have ∞^2 curves in common is given by the fields

$$\varphi = x, \quad \psi = 4y; \qquad \varphi_1 = x^{-3}, \quad \psi_1 = 1,$$

where the equation of second order,

$$xy'' = y',$$

defines ∞^2 curves $y = ax^2 + b$, which are trajectories in both fields.

23. A fortiori $4\infty^1$ curves, or any number of simple systems, *may* belong to two distinct fields. If the four simple systems are given in the form

$$y' = f_i(x, y) \qquad\qquad (i = 1, 2, 3, 4),$$

the field, if it exists, will be *uniquely* defined provided not all the determinants of fourth order in the matrix

$$\| f_i', \quad f_i f_i', \quad f_i^2 f_i', \quad f_i'', \quad 3f_i'^2 - f_i f_i'' \|$$

vanish identically. Here the primes denote complete differentiation with respect to x, so that

$$f' = f_x + ff_y,$$
$$f'' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + ff_y^2.$$

This is the exact formulation of the result stated previously "in general."

24. Consider the simplest of all fields, gravity assumed constant If a cannon ball is projected in any way into the field it describes a vertical parabola. Conversely if every path in an unknown field is a vertical parabola, it follows that the acting force is vertical and constant in intensity. *How many cannon ball experiments would have to be made in order to arrive at this conclusion?*

We confine the discussion for simplicity to a fixed vertical

plane, taken as the xy-plane, so that the equations of motion are

$$\ddot{x} = 0, \qquad \ddot{y} = 1$$

and the trajectories are the ∞^3 parabolas

$$y = ax^2 + bx + c.$$

Suppose first the cannon is kept in one place, say the origin, and the ball is fired in all directions with all initial speeds, giving in all ∞^2 parabolas

$$y = ax^2 + bx.$$

This would not be sufficient to prove that the field is uniform. Another possible field, for example, is

$$\ddot{x} = x^{-5}, \qquad \ddot{y} = yx^{-6}.$$

In fact there are ∞^2 distinct fields each consistent with the given set of ∞^2 parabolas.

The same is true if we confine our geometric experiments to the ∞^2 parabolas $y = ax^2 + c$ found by shooting horizontally from every point in the axis of ordinates with variable initial speed. The differential equation of this family is $xy'' = y'$, precisely the one given at the end of § 22, and so the two forces there given are consistent with the experiments, just as much as ordinary gravity.

If however the shots are fired from all points in the axis of abscissas, with all initial speeds, at the fixed inclination of 45°, producing as trajectories the ∞^2 vertical parabolas whose foci are on the axis of abscissas, *the field must be uniform gravity.* The only possible field is in fact $\ddot{x} = 0$, $\ddot{y} =$ constant.

The same is true if we fix the amount of powder, that is the initial speed, and shoot from every point on the ground (the axis of x), at every angle. This gives ∞^2 parabolas with a common directrix.

As an example of a set of $4\infty^1$ observations that would be sufficient, we mention only the case of shooting from four*

* It may be that three stations are sufficient, but this requires a separate discussion. Two stations would certainly not suffice.

stations on the ground, pointing the cannon at the angle 45°, and using all initial speeds.

25. Consider very briefly the general inverse problem in space of three dimensions. The determination of the magnitude of the force involves the same considerations as in the plane case.

If we are given two trajectories through O in the same direction, the osculating planes must coincide. The force acts in this common plane; its direction is determined by projecting the given space curves orthogonally on this plane, and then using the plane construction described above.

If we are given two trajectories with distinct osculating planes, the initial directions will be necessarily distinct; the force-direction is then determined by the intersection of the osculating planes.

If we are given two trajectories through O in different directions, but with the same osculating plane, the direction of the force is not determined. We need in fact four such curves with the same osculating plane and different directions before the force-direction is determined: the requisite construction is again obtained by orthogonal projections of the curves of the common osculating plane, thus reducing the problem to that considered in the two-dimensional theory (cf. § 18).

§§ 26–27. Tests for a Conservative Field

26. Since the system of trajectories determines the field of force, it ought to be possible to find out from the trajectories, whether the field belongs to any special type, for example, whether the field is central or conservative.

The lines of force are determined geometrically by property I in the plane and property II in space. The field will be central if the lines of force are straight lines passing through a common point.

We now give a number of tests any one of which will distinguish a conservative from a non-conservative force. It is not possible to decide this from the lines of force alone.

1°. First consider the *plane theory*. Here there is for each point a certain conic determined by the trajectories in accordance with property III of § 3 as the locus of the centers of the hyperosculating circles. *For a conservative field (and for no other) this conic is always a rectangular hyperbola.*

2°. In connection with property III (3) of § 8 we have this test: The conic which there appears as the locus of the centers of the focal circles is in the conservative case merely a straight line. That is, the focal circles constructed at any point all have a second point in common.

3°. The focal circles corresponding to two perpendicular directions are, in any field, tangent to each other. In the conservative case the two circles coincide.

4°. In any field two trajectories through a given point O exist whose osculating parabolas have the same given focus. If for one given focus the trajectories are orthogonal at O, this will be true for any given focus. When this is the case for every point O, the force will be conservative.

27. In the *three-dimensional theory*, the lines of force in the conservative case necessarily form a normal congruence; but this is not a sufficient test. All the tests given below are both necessary and sufficient.

1°. First consider property III of § 11. In any field there corresponds to each point O a certain twisted cubic curve Γ. The conservative fields are distinguished by the fact that the cubic Γ is, for every point O, of the rectangular type.*

2°. An interesting kinematic test, connected with the theorem of Thomson-Tait, is the following. If from any point O we shoot with a given speed v_0 in every direction, ∞^2 trajectories will be obtained. If these form a normal congruence (that is admit a set of orthogonal surfaces), the same will necessarily be true for any other speed v_0. *The trajectories starting out from any point with*

* That is, the cubic intersects the plane at infinity in three mutually orthogonal directions. All the quadrics passing through the curve are then of the equilateral type.

a given speed form a normal congruence when, and only when, the field is conservative.

The necessity of this condition is included in the Thomson-Tait theorem discussed in the next chapter. Its sufficiency, of course, requires a separate discussion which is connected with the theory of velocity systems.

3°. In order to make the preceding test purely geometric, it is necessary to have a geometric method of assembling those trajectories which, starting from the same point, correspond to the same initial speed. Such a method is readily found from the fact that the square of the speed varies directly as the radius of curvature and directly as the normal component of the force. The ∞^2 trajectories corresponding to a given speed have circles of curvature intersecting each other at the same point on the line of the force vector; that is, the centers of curvature lie in a plane perpendicular to the direction of the force acting at the given point. In the conservative case, the ∞^2 trajectories so selected form a normal congruence.

4°. Among the ∞^2 trajectories considered there are, for any field, three which admit hyperosculating circles of curvature. The three initial directions thus determined will be mutually orthogonal when and only when the field is conservative.

Only test 1° is directly connected with the set of properties I–IV of page 19. The other three are suggested by the discussion of velocity systems (cf. § 32).

CHAPTER II

NATURAL FAMILIES: THE GEOMETRY OF CONSERVATIVE FIELDS OF FORCE

§ 28. Origin and Application of the Natural Type

28. We now consider the properties of the trajectories generated by conservative fields of force. The total system of trajectories will have the general properties previously considered for an arbitrary field of force, together with the additional properties stated in §§ 26, 27, peculiar to the conservative case.

An entirely new feature presents itself, due to the fact that the differential equations of motion admit an integral of the first order, namely, the energy equation. During any motion of the particle in the given field, the sum of the kinetic and potential energies is constant; thus each motion corresponds to a definite value of the constant h, representing the total energy. The motions may therefore be grouped according to the values of h. Those corresponding to a given value form what may be termed, following Painlevé, a *natural* family.

Thus, in space of two dimensions, the complete system of trajectories for a given conservative field of force consists of ∞^3 curves grouped into ∞^1 natural families, each composed of ∞^2 curves. For example, in the case of ordinary gravity the trajectories are the ∞^3 vertical parabolas (in a given vertical plane), and the natural families are formed by grouping together those parabolas which have the same (horizontal) line as directrix.

In space of three dimensions, the complete system contains ∞^5 trajectories grouped into ∞^1 natural families, each containing ∞^4 curves. Examples are the ∞^4 parabolas with vertical axes whose directrices are situated in a fixed horizontal plane; and the ∞^4 circles orthogonal to a fixed sphere. The simplest example, corresponding to the case of zero force, is the ∞^4 straight lines of space.

This grouping of the trajectories according to the values of the total energy constant, that is, into natural families, is fundamental in most dynamical investigations relating to conservative forces, in particular, those connected with the principle of least action and the developments of Hamilton and Jacobi. From this point of view, dynamical problems relating to the same field of force, but having distinct values of h, are considered as essentially distinct problems. Quoting Darboux: "This restriction is in accordance with the spirit of modern mechanics which attaches less importance to force than to energy, and which permits us to regard as distinct two problems in which the force function or work function is the same, but the total energy is different."

It therefore seems of interest to work out the purely geometric properties of natural families. According to the principle of least action, such a family is made up of the extremals defined by the variation problem

$$\int \sqrt{W + h}\, ds = \text{minimum},$$

that is, the curves which cause the first variation of the integral to vanish. This follows from the fact that the speed v, in the action integral $\int v ds$, is determined by the energy equation

$$v^2 = 2(W + h).$$

Abstractly, a natural family of curves may be defined as one which can be regarded as the totality of extremals connected with a variation problem of the form

$$\int F ds = \text{minimum},$$

where F is any point function, that is, any function of x, y, z in the three-dimensional case.

Such families arise not only in the discussion of trajectories, but also, for example, in the discussion of brachistochrones, catenaries, optical rays, geodesics, and contact transformations.

The brachistochrone problem for a conservative field with any work function W leads to the integral

$$\int dt = \int \frac{ds}{\sqrt{W + h}}.$$

Thus the complete system of brachistochrones is made up of ∞^1 natural families, one for each value of h.

When a homogeneous, flexible, inextensible string is suspended in the conservative field, the forms of equilibrium, which are termed catenaries in the general sense of the word, are obtained by rendering the integral

$$\int (W + h) ds$$

a minimum. Hence here also we have ∞^1 natural families, one for each value of h.*

Consider an isotropic medium in which the index of refraction ν varies arbitrarily from point to point. The paths of light in such a medium, according to Fermat's principle of least time, are determined by minimizing the integral $\int \nu ds$ and hence form a single natural family. This is the most concrete way of defining a natural family.

The connection with the theory of geodesics is obvious. Thus in the two-dimensional case the geodesics of the surface whose squared element of length (first fundamental form) is $\lambda(x, y)(dx^2 + dy^2)$ are found by minimizing the integral $\int \sqrt{\lambda}\, ds$, and hence the representing curves in the x, y plane constitute a natural family. Hence if any surface is represented conformally on a plane, the geodesics are pictured by a natural family of curves in that plane. The extension to more variables is evident:

* The complete systems of ∞^5 brachistochrones and ∞^5 catenaries have geometric properties distinct from each other and from those of the ∞^5 trajectories: no quintuply infinite system of curves can be at the same time the system of trajectories for some field and the system of brachistochrones or catenaries in either the same or a different field. The distinctive properties for an arbitrary field are given in § 107, p. 94. Cf. § 103.

any natural family in any space may be obtained by conformal representation from the geodesics of some other space.*

As a last application we consider the transformations which Sophus Lie has termed the infinitesimal contact transformations of mechanics. In the plane case, such a transformation is defined by a characteristic function of the special form $\Omega(x, y)(1 + y'^2)^{\frac{1}{2}}$ and is characterized by the fact that the lineal elements at each point are converted into the elements of a circle about that point as center. The path curves of every contact transformation of this category form a natural family.

§§ 29–31. Characteristic Properties A and B

29. *Osculating Circles—Property A.*—We now consider the general geometric properties of a natural family in ordinary space, that is, the totality of ∞^4 extremals connected with an integral of the form

$$\int F(x, y, z) \sqrt{1 + y'^2 + z'^2}\, dx. \tag{1}$$

The differential equations of the family are then the corresponding Euler-Langrange equations

$$\begin{aligned} y'' &= (L_y - y'L_x)(1 + y'^2 + z'^2), \\ z'' &= (L_z - z'L_x)(1 + y'^2 + z'^2), \end{aligned} \tag{2}$$

where

$$L = \log F.$$

Of the ∞^4 curves in this family ∞^2 pass through any given point p, one in each direction. Our first result is:

Theorem 1: *The ∞^4 curves in any natural family have this property: the circles which at any point p of space osculate the ∞^2 curves passing through that point, have a second point P in common and thus form a bundle.*

* A natural family on a given surface may be regarded as a family of pseudo-geodesics, that is, one which may be obtained as the conformal picture of the geodesics on some other surface.

This property we shall refer to as *property A*. In the discussion it is convenient to decompose it into these two statements, also relating to the ∞^2 curves through a given point:

(A_1) The osculating planes constructed at the common point form a pencil.

(A_2) The centers of curvature lie in a plane perpendicular to the axis of the pencil of osculating planes.

A proof of the theorem stated is easily obtained by regarding the family as made up of dynamical trajectories. Property A_1 results from the fact that the osculating plane of a trajectory always passes through the force vector. Property A_2 is proved by noting that those trajectories through a given point, which correspond to the same value of the total energy h, are all described with the same initial velocity v_0. The radius of curvature at the initial point is given by the formula

$$r = v_0^2/N,$$

where N denotes the component of the force along the principal normal. Since N is the orthogonal projection of a fixed vector, the locus of its terminal point will be a sphere through the initial point. The conclusion then follows from the fact that r varies inversely as N.

The following analytical discussion has the advantage of answering the converse question which naturally arises: Are there other systems with property A?

The differential equations of any system of ∞^4 space curves, one determined by each lineal element of space, may be assumed in the form

$$(3) \qquad y'' = f(x, y, z, y', z'), \qquad z'' = g(x, y, z, y', z').$$

Property A_1 requires that at each point there shall be a certain direction through which all the osculating planes at that point must pass. Let the direction in question be given by the ratios of three arbitrary point functions

$$(4) \qquad \phi(x, y, z), \qquad \psi(x, y, z), \qquad \chi(x, y, z);$$

then the requisite condition is

$$\frac{z''}{y''} = \frac{\chi - z'\phi}{\psi - y'\phi}. \tag{5}$$

Property A_2 requires that the centers of curvature shall lie in a plane perpendicular to the direction (4); hence

$$\phi X + \psi Y + \chi Z = 1, \tag{6}$$

where X, Y, Z, denote the coordinates of the center relative to axes with the common point as origin. Using the general formulas for the center of curvature, and combining with (5), we find

THEOREM 2: *The differential equations of any system of curves possessing property A are of the form*

$$\begin{aligned} y'' &= (\psi - y'\phi)(1 + y'^2 + z'^2), \\ z'' &= (\chi - z'\phi)(1 + y'^2 + z'^2), \end{aligned} \tag{7}$$

where ϕ, ψ, χ are arbitrary functions of x, y, z. The converse is valid also.

The equations (2) are seen to be included in this form, hence the result certainly holds for our natural systems, as stated in theorem 1.

30. *Hyperosculations—Property B.*—The circles of curvature at a given point, for any system of the form (7), constitute a bundle. We now inquire whether any of these circles correspond to four-point, instead of three-point, contact.

If a twisted curve is to have an hyperosculating circle of curvature at a given point, two conditions must be satisfied, namely,

$$\begin{vmatrix} 1 & y' & z' \\ 0 & y'' & z'' \\ 0 & y''' & z''' \end{vmatrix} = 0, \tag{8}$$

$$\frac{dr}{ds} = 0. \tag{9}$$

The first of these states that the osculating plane has four-point contact with the curve; the second, in which r denotes the radius of curvature, is the condition for the existence of an osculating helix, i. e., one with four-point contact. When both conditions hold the helix is simply the circle of curvature, which then has hypercontact.

Applying these conditions to the curves defined by (7), we find, from (8),

$$(10) \qquad (\psi - y'\phi)\chi' - (\chi - z'\phi)\psi' + (y'\chi - z'\psi)\phi' = 0;$$

and, from (9),

$$(11) \quad (1 + y'^2 + z'^2)\Sigma\phi\phi' - (\phi + y'\psi + z'\chi) \times \left\{ \begin{matrix} (1 + y'^2 + z'^2)\Sigma\phi^2 + \phi' + y'\psi' + z'\chi' \\ - (\phi + y'\psi + z'\chi)^2 \end{matrix} \right\} = 0,$$

where the indicated summations extend over ϕ, ψ, χ and where ϕ', for example, denotes $\phi_x + y'\phi_y + z'\phi_z$.

Since we wish to discuss the ∞^2 curves through a given point, we may simplify our equations considerably by taking the axis of abscissas in the special direction (4). Then, at the selected point, ψ and χ vanish, and the above equations reduce to

$$(10') \qquad y'\chi' - z'\psi' = 0,$$

$$(11') \qquad (y'^2 + z'^2)(\phi' - \phi^2) - (y'\psi' + z'\chi') = 0.$$

Neglecting the trivial solutions for which $y'^2 + z'^2$ vanishes, we may reduce this pair of simultaneous equations to the form

$$(12) \qquad \frac{\psi_x + y'\psi_y + z'\psi_z}{y'} = \frac{\chi_x + y'\chi_y + z'\chi_z}{z'} = \phi_x + y'\phi_y + z'\phi_z - \phi^2.$$

This set of equations for the determination of y', z' is of a familiar type, namely, that arising in the determination of the fixed points of a collineation, and is easily shown to admit three solutions.* Hence

* Of course in special cases some of these may coincide, or the number of solutions may become infinite. The theorem stated is true "in general" in so far as it omits these cases which are definitely assignable.

THEOREM 3: *The curves defined by equations of the form* (7) *are such that through each point there pass three with hyperosculating circles at that point.*

Since the form (7) is characterized by property A, it follows that the existence of three hyperosculating circles in each bundle is a consequence of property A.

We state two further properties, found by considering the conditions (10′) and (11′) separately.

The tangents to those curves of a system (7) *which pass through a given point and there have an hyperosculating plane form a quadric cone.* This cone passes through the special direction (4).

The tangents to those curves which have an osculating helix at the given point form a cubic cone. This cone passes through the special direction (4) and through the minimal directions in the plane normal to that direction.

These properties hold for natural families since they hold for all systems with property A. By comparing (7) with (2), we see that the functions ϕ, ψ, χ in the case of a natural family are

$$\phi = L_x, \qquad \psi = L_y, \qquad \chi = L_z; \tag{13}$$

and hence are connected by the relations

$$\psi_z - \chi_y = 0, \qquad \chi_x - \phi_z = 0, \qquad \phi_y - \psi_x = 0. \tag{14}$$

We now inquire what is the effect of these relations on the directions of the hyperosculating circles. Introducing, for symmetry,

$$X : Y : Z = 1 : y' : z', \tag{15}$$

we may write our equations (12) in the homogeneous form

$$\begin{gathered}\chi_y Y^2 - \psi_z Z^2 + (\chi_z - \psi_y) YZ + \chi_x XY - \psi_x XZ = 0,\\ -\chi_x X^2 + \phi_z Z^2 + \phi_y YZ - \chi_y XY + (\phi_x - \phi^2 - \chi_z) XZ = 0,\\ \psi_x X^2 - \phi_y Y^2 - \phi_z YZ - (\phi_x - \phi^2 - \psi_y) XY + \psi_z XZ = 0.\end{gathered} \tag{16}$$

In virtue of (14), each of the quadric cones (16) is seen* to be

* The condition for such a cone is that the sum of the coefficients of X^2, Y^2, and Z^2 shall vanish.

of the rectangular type. Hence the three generators common to the cones must be mutually orthogonal. This gives

THEOREM 4: *In the case of any natural family the three hyperosculating circles which exist in any bundle are mutually orthogonal.*

We refer to this property as *property B*.

31. The relations (14) are seen to be necessary as well as sufficient for the orthogonality in question. Hence property B is the equivalent of (14), and serves to single out the natural families from the more general class defined by equations of form (7). The latter form was characterized by property A; hence we have our

FUNDAMENTAL THEOREM: *A system of ∞^4 curves, one for each direction at each point of space, will constitute a natural family when, and only when, it possesses properties A and B: that is, the osculating circles at any given point must form a bundle, and the three hyperosculating circles contained in such a bundle must be mutually orthogonal.*

§ 32. GENERAL VELOCITY SYSTEMS

32. The most general system with property A is represented by differential equations of the form

$$(7)\qquad \begin{aligned} y'' &= (\psi - y'\phi)(1 + y'^2 + z'^2), \\ z'' &= (\chi - z'\phi)(1 + y'^2 + z'^2), \end{aligned}$$

and thus involves three arbitrary functions. Only in the case where these functions are the partial derivatives of the same function is the system a natural one. We now point out a dynamical problem that leads to the general type (7): this justifies the term *velocity system* which we hereafter employ to denote any system of this type.

Consider a particle (of unit mass) moving in any field of force, the components of the force being ϕ, ψ, χ. The equations of motion are then

$$\ddot{x} = \phi(x, y, z), \qquad \ddot{y} = \psi(x, y, z), \qquad \ddot{z} = \chi(x, y, z).$$

If the initial position and the initial velocity are given the motion

is determined. If only the initial position and direction of motion are given, the osculating plane will be determined but the radius of curvature r will depend for its value on the initial speed v. Hence, in addition to the usual formula

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2,$$

there must be a formula expressing v^2 in terms of x, y, z, y', z', r. This is furnished by the familiar equation

$$v^2 = rN,$$

where N denotes the (principal) normal component of the force, so that

$$N^2 = \phi^2 + \psi^2 + \chi^2 - \frac{(\phi + y'\psi + z'\chi)^2}{1 + y'^2 + z'^2}.$$

The result may be written in the two (equivalent) forms

$$v^2 = \frac{(\psi - y'\phi)(1 + y'^2 + z'^2)}{y''} = \frac{(\chi - z'\phi)(1 + y'^2 + z'^2)}{z''}.$$

In the actual trajectory v varies from point to point. If now we replace v^2 in this result by some constant, say $1/c$, the resulting equations may be written

$$y'' = c(\chi - y'\phi)(1 + y'^2 + z'^2),$$
$$z'' = c(\chi - z'\phi)(1 + y'^2 + z'^2).$$

The curves satisfying these differential equations—they are not in general trajectories—we define as *velocity curves*. For any field a curve is a velocity curve corresponding to the speed v_0, provided a particle starting from any lineal element of the curve with that speed describes a trajectory osculating the curve. In a given field of force there are ∞^5 trajectories and ∞^5 velocity curves.* If c is given we have ∞^4 velocity curves. In particular

* The properties of a complete system of ∞^5 velocity curves are analogous to, but distinct from, those of a complete system of trajectories. Cf. p. 94.

if c (and hence v) is taken to be unity, our equations become precisely (7).

Any system of ∞^4 *curves possessing property* A, *that is, any system* (7), *may be regarded as the totality of velocity curves corresponding to unit velocity in some (uniquely defined) field of force.*

Only when the field is conservative do the velocity systems for each value of v (or c) become natural systems. The trajectories also are in this case made up of ∞^1 natural families, one for each value of the energy constant h; but the two sets of natural families are distinct. The determination of a velocity system in one conservative field is equivalent to the determination of a trajectory system in another conservative field, and vice versa. We find in fact the following explicit result:

If two conservative fields with work functions W_1 *and* W_2 *satisfy the relation**

$$W_2 = a e^{\frac{2W_1}{v_0^2}} - h,$$

then the ∞^4 *velocity curves for the speed* v_0 *in the first field coincide with the* ∞^4 *trajectories for the constant of energy* h *in the second field.*†

§ 33. Reciprocal Systems

33. With any velocity system S

$$(S) \qquad y'' = (\psi - y'\varphi)(1 + y'^2 + z'^2), \quad z'' = (\chi - z'\varphi)(1 + y'^2 + z'^2)$$

there is connected a definite point transformation T: for in virtue of property A to any point p corresponds a definite point P, the osculating circles constructed at the first point all passing through the second point. The transformation T is explicitly

$$(T) \qquad X = x + \frac{2\varphi}{\varphi^2 + \psi^2 + \chi^2}, \cdots.$$

---

* We note that if W_1 is left unaltered and v_0 varied, W_2 takes quite distinct forms. The ∞^1 velocity systems in a given field do not constitute the complete system of ∞^5 trajectories in any field whatever.

† It is seen that the two fields have the same equipotential surfaces and therefore the same lines of force. (Central fields therefore correspond to central fields.)

It is thus entirely general. To an arbitrary transformation* corresponds a definite velocity system. In particular, to the inverse transformation T^{-1} there corresponds a certain system S', which we define as reciprocal to S.

Hence to a general† velocity system S, that is, any system possessing property A, there corresponds a definite reciprocal velocity system S'. The osculating circles of those curves of system S which pass through any point p are at the corresponding point P the osculating circles of the curves of the system S' passing through P.

Consider the bundle of circles determined by two corresponding points p and P. We know that three of these circles have hypercontact with S-curves at p, and three have hypercontact with S'-curves at P. It is not obvious that the circles so obtained really coincide. Omitting the rather long proof, we merely state the result.

Reciprocal velocity systems have the same hyperosculating circles: the three circles hyperosculating curves of the given system S at any point p also hyperosculate curves of the reciprocal system S' at the corresponding point P.

It follows at once that if S possesses property B (that is mutually orthogonal hyperosculating circles) the same will be true of S'. This means that whenever system S is natural so is S'.

The reciprocal of a natural family is always a natural family.

We may restate this in optical terms as follows: With any isotropic medium, defined by its index of refraction $\nu(x, y, z)$, there is connected a certain *reciprocal medium* with an index of refraction $\bar{\nu}(x, y, z)$: the rays of light in this second medium, namely, the extremals of

$$\int \bar{\nu}(x, y, z)ds = \text{minimum},$$

form the system reciprocal to that formed by the rays of light

* It may even degenerate but must not be merely the identical transformation. We however exclude systems with degenerate T's from the rest of the discussion: we assume that the jacobian does not vanish, so that the inverse transformation exists.

† See preceding footnote.

in the given medium, namely, the extremals of

$$\int \bar{\nu}(x, y, z)ds = \text{minimum}.$$

The actual calculation of $\bar{\nu}$ from ν requires only operations that are *performable* in the Lie sense, namely, eliminations and differentiations. See *Transactions of the American Mathematical Society*, volume 10 (1909), page 213.

§ 34. Character of the Transformation T

34. The transformation T (from point p to point P) associated with the most general system possessing property A is, as we have seen, entirely arbitrary. The question arises what is the peculiarity of T if the given system is of the natural type. The answer to this will furnish an equivalent of property B, and will thus make it possible to characterize natural families without introducing hyperosculating circles.

The problem is to describe geometrically the class of transformations of the form

$$X = x + \frac{2L_x}{L_x^2 + L_y^2 + L_z^2}, \quad Y = y + \frac{2L_y}{L_x^2 + L_y^2 + L_z^2},$$
$$Z = z + \frac{2L_z}{L_x^2 + L_y^2 + L_z^2},$$

depending on one arbitrary function L of x, y, z, instead of three independent functions required in a general point transformation,

$$X = \Phi(x, y, z), \qquad Y = \Psi(x, y, z). \qquad Z = \mathrm{X}(x, y, z).$$

For a general (analytic) point transformation the bundle of lineal elements at any point is converted linearly into the bundle at the corresponding point. Are there any elements which go over into parallel elements? It is well known that there are three. If in particular these three elements are mutually perpendicular (for every point of space), we obtain a certain category of transformations which may be termed Darboux* transforma-

* See *Proceedings of the London Mathematical Society*, 1900.

tions or deformations. They are analytically of the form

$$X = f_x, \qquad Y = f_y, \qquad Z = f_z,$$

involving one arbitrary function. Obviously this is not the class we desire.

We next ask whether in the general transformation there are any elements at a given point p each of which is turned into a *cocircular* element at the corresponding point P. This is, in a way, a case correlative to the Darboux case: for whether two elements in space are parallel or cocircular they have in common the properties that they are coplanar and equally inclined to the line pP joining their points. It is found that there are always three such elements at any point. If we require these to be mutually orthogonal, we obtain precisely the transformations connected with natural families.

A system of ∞^4 space curves possessing property A will form a natural family when and only when the associated transformation T (from point p to point P) has the following property: the three lineal elements at p each of which is converted into a cocircular element at P are mutually orthogonal.

We have thus obtained an equivalent for property B. It may be shown synthetically that the three directions just described (cocircular elements) always coincide with the directions of the hyperosculating circles. The orthogonality of the one triple amounts to the same thing as the orthogonality of the other.

It may be remarked that the class of transformations connected with all natural systems do not form a group. It is obvious however that the inverse of any member of the class is contained in that class. *This is the essence of the law of reciprocity for natural systems, previously obtained by a different method.*

§§ 35–44. The Converse of Thomson and Tait's Theorem

35. It is well known that if straight lines are drawn orthogonal to any given surface they will necessarily be orthogonal to an

infinitude of surfaces (namely the surfaces parallel to the given surface). Thomson and Tait in their Natural Philosophy showed that this property of the ∞^4 straight lines of space holds for the ∞^4 trajectories described in any conservative field with the same total energy, that is, for any natural family. The writer has proved that no other families of curves have the property: it is entirely characteristic of the natural type.* We first state the original theorem in connection with the general theory of the calculus of variations, and then take up the converse theorem. Later a second converse question is discussed.

35′. *Thomson and Tait's Theorem.*—We have seen that a natural family of curves in space may be regarded as the totality of extremals of a variation problem of the particular form

$$J = \int F(x, y, z)ds, \tag{1}$$

where F is a point function, ds is the element of length

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{1 + y'^2 + z'^2}\,dx,$$

and the integral is taken between fixed end points.

It is easily shown that for integrals of this form,† and for no others, the relation of *transversality*, in the sense of the calculus of variations, amounts merely to *orthogonality*. This suffices to distinguish our type among variation problems of the general form

$$\int f(x, y, z, y' z')dx. \tag{2}$$

But of course it does not serve as a complete geometric test for a natural family. What is the geometric character of the systems of ∞^4 extremals connected with any variation problem(2)? This is an unsolved question in the calculus of variations.‡

* At least in the case of space of three dimensions. Cf. *Trans. Amer. Math. Soc.*, vol. 11 (1910), pp. 121–140.

† Cf. Bolza, Variationsrechnung, p. 691; also p. 146 for the two-dimensional problem due to Hedrick.

‡ See the author's paper, "Systems of extremals in the calculus of variations," *Bull. Amer. Math. Soc.*, vol. 13 (1908), pp. 289–292.

We are concerned here only with the integrals of special form J, defining natural families. Applying Kneser's fundamental theorem on transversals,* we have this well-known result: If from the points of any surface Σ we construct the extremals orthogonal to the surface, and on each lay off an arc so that the integral J takes some constant value, then the locus of the end points is a surface which is also orthogonal to the extremals.

36. This is known as the *theorem of Thomson and Tait.* It was obtained by them in connection with the dynamics of a particle moving in a conservative field—the first interpretation of a natural family considered in § 28. Here $F(x, y, z)$ represents the speed v, as determined by the energy equation

$$v^2 = 2(W + h),$$

where W denotes the work function (negative potential), and the mass is assumed to be unity. Of course h has a fixed value. We quote the original statement of the theorem:

"If from all points of an arbitrary surface particles not mutually influencing one another be projected normally with the proper velocities [so as to make the sum of the kinetic and potential energies have a given value]; particles which they reach with equal actions lie on a surface cutting the paths at right angles."

The integral J, in this case, represents the action

$$\int v\,ds = \int \sqrt{2(W + h)}\,ds.$$

The ∞^1 surfaces cutting the curves orthogonally thus appear as surfaces of equal action.

The corresponding statement for brachistochrones is sometimes called the *theorem of Bertrand*:† From the points of any surface draw the brachistochrones normal to the surface and on each lay off lengths so that the time of transit is equal to a given quantity; then the locus of the end points will be another surface orthogonal

* Bolza, pp. 131 and 691.

† Cf. Routh, Dynamics of a Particle (1898), p. 376. According to Appell, Mécanique rationelle, vol. 1 (1909), p. 466, this result was indicated by Euler.

to the brachistochrones. Here the integral J represents the time

$$\int dt = \int \frac{ds}{v} = \int \frac{ds}{\sqrt{2(W+h)}},$$

so that the orthogonal surfaces appear as surfaces of equal time.

Corresponding statements may be made, of course, for the other interpretations leading to natural families. The most concrete aspect is obtained by using the language of optics. Here the integrand function is simply the index of refraction $\nu(x, y, z)$, varying from point to point in any (isotropic) medium, and the integral $\int \nu ds$ is proportional to the time. The paths of light in such a medium form a (single) natural family, and every natural family may be obtained in this way. The ∞^2 rays (in general curved) starting out normally from anys urface admit ∞^1 orthogonal surfaces. These present themselves as surfaces of equal time. We shall describe them as a *set of wave fronts or wave surfaces.*

37. *The geometric part of the theorem of Thomson and Tait may be stated as follows: In any natural family of ∞^4 space curves, the ∞^2 curves which meet any surface orthogonally always form a normal congruence.*

Is this geometric property, which we shall refer to as the *Thomson-Tait property,* characteristic? This is in fact the case. We shall prove, namely, the following

CONVERSE THEOREM. *If a quadruply infinite system of curves in space is such that ∞^2 of the curves meet an arbitrarily given surface orthogonally* and always form a normal congruence (that is, admit an infinitude of orthogonal surfaces), then the system is of the natural type, that is, it may be identified with the extremal system belonging to an integral of the form $\int F(x, y, z)ds$.*

38. The result is simple but the proof is rather long. We give the essential steps.

Consider an arbitrary quadruply infinite system of curves in

* This means the same as requiring that one curve of the system passes through each point of space in each direction.

space, assuming that one passes through each point in each direction. Such a system may be defined by a pair of differential equations of the second order

$$(1) \qquad y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z'),$$

where F and G are uniform functions which we assume to be analytic in the five arguments. Denoting the initial values of x, y, z, y', z', which may be taken at random, by x, y, z, p, q respectively, and employing X, Y, Z as current coordinates, we may write the solutions of (1) in the form

$$(2) \qquad \begin{aligned} Y &= y + p(X - x) + \tfrac{1}{2}F(X - x)^2 + \tfrac{1}{6}M(X - x)^3 + \cdots, \\ Z &= z + q(X - x) + \tfrac{1}{2}G(X - x)^2 + \tfrac{1}{6}N(X - x)^3 + \cdots. \end{aligned}$$

Here F and G are expressed as functions of x, y, z, p, q; and M and N, found by differentiating (1), are given by

$$(3) \qquad \begin{aligned} M &\equiv F_x + pF_y + qF_z + FF_p + GF_q, \\ N &\equiv G_x + pG_y + qG_z + FG_p + GG_q. \end{aligned}$$

The terms of higher order will not be needed in our discussion. Equations (2) involve five arbitrary parameters but of course represent only ∞^4 curves.

Consider now an arbitrary surface Σ

$$(4) \qquad z = f(x, y).$$

At each point of this surface and normal to it a definite curve of the given family (1) may be constructed. A certain congruence will thus be determined. We wish to express the condition that this shall be of the normal type, that is, that the ∞^2 curves shall admit a family of orthogonal surfaces.

The direction normal to the surface Σ at any point is given by

$$1 : p : q = f_x : f_y : -1,$$

so that

$$(5) \qquad p = P(x, y), \qquad q = Q(x, y),$$

where

$$(5') \qquad P \equiv f_y/f_x, \qquad Q \equiv -1/f_x.$$

These functions are connected by the relation

$$(5'') \qquad PQ_x - QP_x - Q_y \equiv 0.$$

The equations of the ∞^2 curves corresponding to the given initial conditions may now be written

$$(6) \qquad \begin{aligned} X &= x + t, \\ Y &= y + Pt + \tfrac{1}{2}\bar{F}t^2 + \tfrac{1}{6}\bar{M}t^3 + \cdots, \\ Z &= f + Qt + \tfrac{1}{2}\bar{G}t^2 + \tfrac{1}{6}\bar{N}t^3 + \cdots, \end{aligned}$$

where t takes the place of $X - x$ in (2), and where the bars indicate that the substitution (4), (5) has been carried out, so that, for example,

$$(7) \qquad \bar{F}(x, y) \equiv F(x, y, f, P, Q).$$

The coefficients of the powers of t in (6) are thus functions of the two parameters x, y.

The general condition for a normal congruence given in parametric form is*

$$(8) \qquad (Y'XY) - (Z'ZX) + Y'(Z'YZ) - Z'(Y'YZ) = 0,$$

where the parentheses denote jacobians taken with respect to t, x, y, and Y', Z' denote the derivatives of Y, Z respectively with respect to t.

Expanding (8) in powers of t in the form

$$(9) \qquad \Omega_0 + \Omega_1 t + \Omega_2 t^2 + \cdots,$$

we find that Ω_0 vanishes in consequence of (5''). This is as it should be, since our ∞^2 curves are orthogonal to Σ by construction.

The terms containing the first power of t give

$$(10) \qquad \begin{aligned} &(1 + P^2 + Q^2)(P\bar{G}_x - Q\bar{F}_x - \bar{G}_y) \\ &\qquad + 2\bar{F}\{(P^2+Q^2)(Q_x - PQ_y) + 2\bar{G}fPP_y - (P^2+Q^2)P_x\} = 0. \end{aligned}$$

* We may also use the convenient form due to Beltrami. Cf. Bianchi-Lukat, Differentialgeometrie, p. 340.

From (6′) we find

$$\overline{F}_x = F_x + F_z f_x + F_p P_x + F_q Q_x,$$

with corresponding results for $\overline{G}_x$ and $\overline{G}_y$. Substituting these values, and observing from (5) and (5′) that

$$f_x = -1/Q, \qquad f_y = -P/Q, \qquad Q_y = PQ_x - QP_x,$$

we may reduce (10) to

$$\begin{aligned} &2F\{PQP_x+Q^2Q_x\}+2G\{PP_y-(P^2+Q^2)P_x\}-(1+P^2+Q^2)\\ (10') \quad &\times\{QF_x-F_z-PG_x+G_y+(QF_p-QG_q-PG_p)P_x\\ &\qquad +G_pP_y+QF_qQ_x\}=0. \end{aligned}$$

This is then a necessary condition in order that the ∞^2 curves belonging to the quadruply infinite system (1) and orthogonal to the surface (4) shall form a normal congruence. The result is to hold in virtue of (4) and (5).

It is of course not a sufficient condition. It merely expresses the fact that the curves orthogonal to Σ are also orthogonal to some consecutive surface, that is, that the congruence is approximately normal to the first degree.

Our main problem is to find all systems (1) which have the orthogonality property with respect to *every* base surface Σ. It is then necessary that (10′) should be true for an arbitrary function $f(x, y)$. The function can be so selected that for any chosen values of x and y the quantities f, P, Q, P_x, P_y, Q_x, shall take arbitrary numerical values; for the only relation to be fulfilled is (5″) and this merely determines Q_y. The condition (10′) must therefore hold identically. Arranging it in the form

$$(10'') \quad (1 + P^2 + Q^2)C_0 + QC_1Q_x + C_2P_y - C_3P_x = 0,$$

and equating coefficients to zero, we find

$$\begin{aligned} &C_0 \equiv qF_x - F_z - pG_x + G_y = 0,\\ &C_1 \equiv (1 + p^2 + q^2)F_q - 2qF = 0,\\ (11) \quad &C_2 \equiv (1 + p^2 + q^2)G_p - 2pG = 0,\\ &C_3 \equiv (1 + p^2 + q^2)\{qG_q + pG_p - qF_p\}\\ &\qquad\qquad + 2pqF - 2(p^2 + q^2)G = 0. \end{aligned}$$

Integration of the second and third of these partial differential equations gives

$$F = f_1(p, x, y, z)(1 + p^2 + q^2), \qquad G = g_1(q, x, y, z)(1 + p^2 + q^2),$$

where f_1 and g_1 denote unknown functions of the four arguments indicated. Substituting these values in the fourth equation, we find $f_{1p} = g_{1q}$, and therefore

$$f_1 = \psi - p\phi, \qquad g_1 = \chi - q\phi,$$

where ϕ, ψ, χ are functions of x, y, z only. The general solution of the last three equations of the set (11) is therefore

$$(12) \quad F = (\psi - p\phi)(1 + p^2 + q^2), \qquad G = (\chi - q\phi)(1 + p^2 + q^2).$$

We have still to satisfy the first equation of (11), which now reduces to

$$(13) \qquad \psi_z - \chi_y + p(\chi_x - \phi_z) + q(\phi_y - \psi_x) = 0.$$

The functions ϕ, ψ, χ must therefore satisfy the equations

$$(13') \quad \psi_z - \chi_y = 0, \qquad \chi_x - \phi_z = 0, \qquad \phi_y - \psi_x = 0$$

and hence are expressible as the derivatives of a single function in the form

$$(13'') \qquad \phi = L_x, \qquad \psi = L_y, \qquad \chi = L_z.$$

The solutions of the set (11) are therefore

$$(14) \qquad \begin{aligned} F &= (L_y - pL_x)(1 + p^2 + q^2), \\ G &= (L_z - pL_x)(1 + p^2 + q^2), \end{aligned}$$

involving an arbitrary function L of x, y, z. The resulting system (1) is thus recognized to be a natural family. This gives our fundamental converse theorem.

39. In the above discussion use has been made, not of the complete condition for a normal congruence, but only of condition (10′) derived from the terms of the first order in t. We may therefore state a stronger converse result as follows:

The only systems of ∞^4 curves which have the property that the curves orthogonal to any surface are always orthogonal to some infinitesimally adjacent surface are those of the natural type.

If a congruence of curves meets two neighboring surfaces orthogonally it need not meet ∞^1 surfaces orthogonally, and therefore it approximates to, but need not coincide with, a normal congruence. The above theorem shows however that if the weak requirement of approximate normal character be imposed on *all* the congruences obtained from the given quadruply infinite system, they will *all* be exactly normal.

40. We may further strengthen our theorem by demanding the orthogonality property for some instead of all surfaces. Our fundamental equations (11) resulted from the fact that x, y, z, f, P, Q, P_x, P_y, Q_x might receive arbitrary numerical values. It will therefore be sufficient to take a manifold of surfaces sufficiently large to leave these quantities, or the equivalent quantities

$$(15) \qquad x, \quad y, \quad z, \quad z_x, \quad z_y, \quad z_{xx}, \quad z_{xy}, \quad z_{yy},$$

unrestricted. Since these quantities define a differential surface element of the second order, we may state the result as follows:

The converse theorem remains valid if, instead of considering all base surfaces, we employ a manifold of surfaces sufficiently large to include all the ∞^8 possible differential elements of the second order.

41. The Thomson-Tait theorem holds of course even when the base Σ shrinks to a curve or a point: there will still be a normal congruence orthogonal to the curve or point (in the latter case orthogonality means simply passage through the point). We state a number of results obtained in this connection.

If for an arbitrary curve as base the corresponding ∞^2 orthogonal curves of a given quadruply infinite system always form a normal congruence, the given system is necessarily natural.

If we require each of the congruences here considered to be of approximately normal character, a more general type of system

is obtained, namely the *velocity type* of § 32. The velocity type is thus characterized by the fact that those curves of the system which meet an arbitrary curve orthogonally are orthogonal to some infinitesimally adjacent (of course tubular) surface. We may even restrict ourselves to the case where the base is a curve of the given system, or the case where it is any straight line.

42. Suppose next that the base is an arbitrary point. Are natural families the only families of ∞^4 curves such that the ∞^2 curves passing through any point form a normal congruence? A discussion shows that this is not the case. There exist families not of the natural type, for example, that defined by the differential equations

$$y'' = y'^2, \qquad z'' = 0,$$

with the restricted property stated. To find all such systems would be a rather difficult, but certainly an interesting, undertaking. The result would of course include the natural type as a special case.

43. It will not however be the velocity type. It may be shown in fact that the only velocity systems for which the curves passing through an arbitrary point constitute always a normal congruence are those of the natural type. Recalling the fact that the velocity type is characterized by property A, we may give a new characterization of the natural type as follows:

Natural families are the only quadruply infinite systems of curves in space such that the ∞^2 curves through an arbitrary point admit an infinitude of orthogonal surfaces, and such that the osculating circles constructed at the common point form a bundle.

44. It may also be shown that if for every point and every straight line as base the corresponding congruence is normal, the system will be natural. To have a velocity system it is sufficient to demand that the congruence corresponding to an arbitrary straight line shall be approximately normal. To have a natural system it is sufficient to demand approximate normality for the congruences corresponding to arbitrary straight lines and planes.

§§ 45–53. Wave Propagation in an Isotropic Medium: Properties of Wave Sets

45. The optical interpretation of a natural family and the Thomson-Tait property suggest certain sets of surfaces which we shall now study.

Consider a given medium defined by its index of refraction $\nu(x, y, z)$ given as a function of position. The rays (in general curved lines) are the ∞^4 extremals of

$$\int \nu(x, y, z)ds = \text{minimum}; \tag{1}$$

they form the natural family, whose differential equations are

$$\begin{aligned} y'' &= (L_y - y'L_x)(1 + y'^2 + z'^2), \\ z'' &= (L_z - z'L_x)(1 + y'^2 + z'^2), \end{aligned} \tag{2}$$

where

$$L \equiv \log \nu. \tag{2'}$$

The ∞^2 rays orthogonal to any selected surface Σ form a normal congruence, that is, are orthogonal to a set of ∞^1 surfaces. A disturbance originating in the medium on the surface Σ will be propagated in the medium through this set of surfaces, which we term a *set of wave fronts*. In the given medium an arbitrary surface belongs to one and only one of these wave sets. A single surface is thus of arbitrary character, but the sets of surfaces

$$f(x, y, z) = \text{constant} \tag{3}$$

that may be wave sets are restricted by the Hamilton-Jacobi equation

$$f_x^2 + f_y^2 + f_z^2 = \nu^2. \tag{4}$$

The given medium defines also a certain set of *level surfaces*

$$\nu(x, y, z) = \text{constant}.$$

This, it should be noticed, is not usually a wave set—the only exception arising when the level surfaces are parallel. For a given

medium the number of wave sets is ∞^∞, since there is one for each surface. Each of these sets is cut by the level surfaces in the equidistant curves of the wave set; that is, along any one of these curves the distance between consecutive wave surfaces remains the same.*

46. A single set of wave fronts has no geometric peculiarity. That is, given any set of surfaces $f(x, y, z) =$ constant, it will always be possible to find a medium in which that set will serve as a wave set. In fact there are ∞^∞ such media. For in equation (4), the given function f, without altering the given surfaces, may be replaced by an arbitrary function $\Omega(f)$ of itself, and this gives ∞^∞ distinct values for ν.

When will two sets of wave fronts be consistent? Two arbitrary sets of surfaces $f =$ constant, $f_1 =$ constant cannot usually be regarded as wave sets in any single medium. The requisite condition is

$$\frac{f_x^2 + f_y^2 + f_z^2}{f_{1x}^2 + f_{1y}^2 + f_{1z}^2} = \frac{\Omega(f)}{\Omega_1(f_1)},$$

where Ω, Ω_1 may be any functions. An equivalent condition is that it must be possible to chose parameters for the two sets in such a way that

$$\frac{df}{dn} = \frac{df_1}{dn_1},$$

where dn and dn_1 denote the normal distance between consecutive surfaces.

47. But a clearer answer may be given in terms of the geometric properties A and B. If a set of surfaces is to be a wave set, the ∞^2 orthogonal curves must be members of the natural family of ∞^4 rays. If two sets of wave fronts are given, we have then two congruences of curves. The question then is, when can two normal congruences of curves be regarded as belonging to a natural family?

* This follows immediately from (4). It is to be remarked, however, that this property is not characteristic of wave sets.

Take any point p in space, and consider the two curves, one from each of the congruences, passing through it. The circles of curvature at p must intersect again at some point P (by property A). This condition makes sure that the two congruences belong to some velocity system. If now this is to be a natural system, we must also add property B or rather, since no hyperosculating circles are directly defined, the equivalent restriction (see page 47) relating to the transformation from p to P. The final answer may then be given as follows:

Two sets of wave surfaces belong to the same optical medium when and only when they satisfy the following geometric conditions:

(A') *At any point p of space the circles of curvature of the orthogonal trajectories of the two sets of surfaces, passing through that point, intersect again at some point P.*

(B') *The point transformation from p to P has the property that the three lineal elements of p each of which corresponds to a cocircular element at P are mutually orthogonal.*

48. Two sets of surfaces taken at random will not belong, as wave sets, to any medium. On the other hand, as we have said, one set belongs to ∞^∞ distinct media. The question then arises, just what will uniquely determine a medium.

A natural family is uniquely determined if we are given one set of wave fronts and a single extra trajectory. This means a trajectory not belonging to the congruence defined as the orthogonal trajectories of the wave set.

49. The *extra curve* however cannot be taken at random; it must be related in a certain way to the wave set. If the wave set is $f(x, y, z) =$ constant, then the condition on the curve is that it satisfy the Monge equation of second order

$$(3)\qquad \begin{vmatrix} 2\Delta y'' - (1 + y'^2 + z'^2)(\Delta_y - y'\Delta_x) & f_y - y'f_x \\ 2\Delta z'' - (1 + y'^2 + z'^2)(\Delta_z - z'\Delta_x) & f_z - z'f_x \end{vmatrix} = 0,$$

where

$$(3')\qquad \Delta = f_x^2 + f_y^2 + f_z^2.$$

can have in common is ∞^2 (one through each point of space). If two media have that many in common, it is easily shown that the resulting congruence is necessarily normal. Any normal congruence can be obtained in this way, for, as stated above, it belongs, not only to two, but to ∞^∞ distinct media.

53. We mention only one special problem: the determination of those media in which disturbances are propagated by Lamé families of surfaces; that is, every wave set is to be of the Lamé type (thus forming part of a triply orthogonal family of surfaces). The index of refraction is found to vary inversely as the *power* of the point with respect to a fixed sphere; the rays then are the ∞^4 circles orthogonal to that sphere. Since the radius of the sphere may be zero, real, or imaginary, these media yield well known interpretations of parabolic, hyperbolic, and elliptic geometries. (See *Transactions of the American Mathematical Society,* volume 12 (1911), pages 70–74.)

§§ 54–61. A Second Converse Problem Connected with the Thomson-Tait Theorem

54. Consider the general conservative field, defined by its work function $W(x, y, z)$. With any motion of the particle there is associated a definite value of the constant of total energy

$$\tfrac{1}{2}v^2 - W = h.$$

If h is not assigned the complete system of trajectories is made up of ∞^5 curves.

Consider now an arbitrary surface, which we term the base surface,

$$(\Sigma) \qquad z = f(x, y).$$

From each of its points we may draw normal to the surface ∞^1 trajectories since the initial value of the speed v is arbitrary. We thus have in all ∞^3 trajectories normal to Σ. In order to have a congruence we must assign the value of v at each point of Σ, that is, we must give a law of distribution of the initial speed. The question arises: What form of law will make the corresponding

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congruence a normal congruence? Of course for any law the congruence will be orthogonal to the base surface, but usually it admits no other orthogonal surfaces.

The Thomson-Tait theorem (in its complete dynamical form) gives *one* such law: it states that if the initial speed is selected so as to make h have the same value at all the points of Σ, the congruence will be normal. It thus gives a plan for constructing ∞^1 normal congruences for a given base, one for each value of h. We shall refer to any one of these as "constructed according to the Thomson-Tait law."

Is this the only answer to our question? If ∞^2 trajectories are drawn orthogonal to Σ and if they form a normal congruence, does it follow that the distribution of values of the initial speed is precisely such that the sum of the kinetic and potential energies has the same value at all points of Σ?

The requisite discussion is not simple. We shall merely state the results we have obtained.

55. The answer to our question is "in general" in the affirmative. The first converse theorem, discussed in § 37, is true without exception. The present is true with exceptions—which may be definitely limited.

For a "general" base surface Σ in a given conservative field of force, the only congruences, formed by ∞^2 trajectories orthogonal to Σ (one drawn at each point), which admit an infinitude of orthogonal surfaces, are those constructed according to the Thomson-Tait law (so that the total energy has a constant value).

56. To make this precise we must of course limit the class of *exceptional surfaces* connected with a given field. These appear in the analytical discussion as the solutions of a certain partial differential equation of the second order*

$$\begin{vmatrix} W_x + pW_y & \overline{W}_x + p\overline{W}_y \\ W_x + qW_z & \overline{W}_x + q\overline{W}_z \end{vmatrix} = 0,$$

* The expanded result is of the form

$$P_1 r + P_2 s + P_3 t + P_4 = 0,$$

where r, s, t denote the derivatives of second order of $z = f(x, y)$.

where W is the given work function, and

$$\overline{W} \equiv \frac{pW_x + qW_y - W_z}{\sqrt{1 + p^2 + q^2}}.$$

This differential equation defines a class of surfaces which is seen to depend only on the equipotential surfaces

$$W(x, y, z) = \text{constant}.$$

The result may be put into geometric form and stated as follows:

The only surfaces Σ which may be exceptional in the theorem of § 55 *(that is, which may give rise to normal congruences not included in the Thomson-Tait law) are those with this property: along each of the equipotential lines* of the surface the component of the acting force normal to the surface is constant.*

57. Observe that it is not stated that the surfaces described, which exist in any field, actually give rise to additional normal congruences. To understand the situation more precisely, it is necessary to observe that in the analytic discussion the condition for a normal congruence is developed in the form

$$t\,\Omega_1 + t^2\Omega_2 + \cdots = 0,$$

where t is the parameter which varies along the curve, starting with the value zero on the surface Σ, and the coefficients Ω are functions of the two parameters defining the initial points on Σ. By assumption the congruence is orthogonal to Σ, so the term Ω_0, independent of t, will not appear. For a normal congruence all the coefficients Ω must vanish. If only a certain number vanish the congruence may be described as *approximately normal* (the approximation being of degree n if $\Omega_1 = \Omega_2 \cdots = \Omega_n = 0$): the curves are then orthogonal not only to Σ but also to one or more (infinitesimally) adjacent surfaces.

58. If now we impose on the congruence of trajectories normal to Σ the condition $\Omega_1 = 0$, we find that this may be fulfilled for

* The equipotential lines of any surface are the lines cut out by the equipotential surfaces $W = \text{const}$.

any surface: the restriction is merely on the law of initial speed and means that the total energy must be the same, not necessarily over the entire surface, but along each equipotential line of the surface.*

59. If we further impose the condition $\Omega_2 = 0$, then for a "general surface" the law of speed must be the Thomson-Tait law, but for an "exceptional surface" the law is the more general one just stated.

60. The discussion of the higher conditions $\Omega_3 = 0$, etc., we have not completed. It is therefore not known precisely in which cases normal congruences (in the exact sense) may arise. For central and parallel fields it may be shown that the exceptional surfaces† actually give rise to normal congruences (in addition to those included in the Thomson-Tait theory): for such fields the vanishing of the higher coefficients follows from the vanishing of the first two.

61. The principal results of the converse problem may be formulated as follows:

If ∞^2 trajectories (of a conservative field), meeting a surface Σ orthogonally, are also orthogonal to an infinitesimally adjacent surface, then the total energy along each equipotential line of Σ is constant.

If ∞^2 trajectories, selected from the complete system of ∞^5, form a normal congruence, then in general they will all belong to the same natural family (that is, the total energy will be the same for all the curves); except possibly when the ∞^1 orthogonal surfaces‡ are exceptional in the sense defined in § 56 *(the additional congruences then and only then are normal to at least the second degree of approximation).*

Normal congruences not of the Thomson-Tait type (that is, not

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* If, in particular, the surface is one of the equipotential surfaces, the distribution of speed is thus entirely arbitrary.

† In the case of ordinary constant gravity the exceptional surfaces are those termed *moulure* surfaces by Monge: they are generated by rolling the plane of any plane curve about a vertical cylinder of arbitrary cross section.

‡ If one of these surfaces is exceptional, all will be.

selected from within a natural family) actually arise for central and parallel fields.

§§ 62–67. Geometric Formulation of Some Curious Optical Properties

62. In Thomson and Tait's Natural Philosophy* the characteristic function of Hamilton is applied to the motion of a particle in a conservative field of force, and certain results are obtained which we shall try to restate as purely geometric properties of a natural family of trajectories. To what extent these properties are characteristic is not settled. We quote the principal passages referred to.

"Let two stations, O and O', be chosen. Let a shot be fired with a stated velocity, V, from O, in such a direction as to pass through O'. There may clearly be more than one natural path by which this may be done; but, generally speaking, when one such path is chosen, no other, not considerably diverging from it, can be found; and any infinitely small deviation in the line of fire from O, will cause the bullet to pass infinitely near to, but not through, O'. Now let a circle, with infinitely small radius r, be described round O as center, in a plane perpendicular to the line of fire from this point, and let—all with infinitely nearly the same velocity, but fulfilling the condition that the sum of the potential and kinetic energies is the same as that of the shot from O —bullets be fired from all points of this circle, all directed infinitely nearly parallel to the line of fire from O, but each precisely so as to pass through O'. Let a target be held at an infinitely small distance, a', beyond O', in a plane perpendicular to the line of the shot reaching it from O. The bullets fired from the circumference of the circle round O, will, after passing through O', strike this target in the circumference of an exceedingly small ellipse, each with a velocity (corresponding of course to its position, under the law of energy) differing infinitely little from V', the common velocity with which they pass through O'. Let now a circle, equal to the former, be described round O',

* Part I (Cambridge, 1903), pp. 355–359.

in the plane perpendicular to the central path through O', and let bullets be fired from points in its circumference, each with the proper velocity, and in such a direction infinitely nearly parallel to the central path as to make it pass through O. These bullets, if a target is held to receive them perpendicularly at a distance $a = a'V/V'$, beyond O, will strike it along the circumference of an ellipse equal to the former and placed in a " corresponding " position; and the points struck by the individual bullets will correspond; according to the following law of " correspondence ":—Let P and P' be points of the first and second circles, and Q and Q' the points of the first and second targets which bullets from them strike; then if P' be in a plane containing the central path through O' and the position which Q would take if its ellipse were made circular by a pure strain; Q and Q' are similarly situated on the two ellipses."

63. The second passage is as follows: " The most obvious optical application of this remarkable result is, that in the use of any optical apparatus whatever, if the eye and the object be interchanged without altering the position of the instrument, the magnifying power is unaltered." . . . " Let the points O and O' be the optic centers of the eyes of two persons looking at one another through any set of lenses, prisms, or transparent media arranged in any way between them. If their pupils are of equal size in reality, they will be seen as similar ellipses of equal apparent dimensions by the two observers. Here the imagined particles of light, projected from the circumference of the pupil of either eye, are substituted for the projectiles from the circumference of either circle, and the retina of the other eye takes the place of the target receiving them, in the general kinetic statement."*

* This fact and many other applications are included in the following general proposition. " The rate of increase of any one component momentum, corresponding to any one of the coordinates, per unit of increase of any other coordinate, is equal to the rate of increase of the component momentum corresponding to the latter per unit increase or dimension of the former coordinate, according as the two coordinates chosen belong to one configuration of the system, or one of them belongs to the initial configuration and the other to the final."

64. The statement in the first passage is not purely geometric; for it involves not only the curves described, but also the speeds V and V' at the points O and O'. We therefore try to formulate the part of the theorem which is really geometric.

We have a natural family made up of ∞^4 curves in space, one for each initial lineal element (point and direction) of space. Select any one of these curves c and any two points O and O' upon it. Construct the planes p and p' normal to this curve at O and O'.

For each direction through O, a curve of our family is determined; this strikes the plane p' at a definite point. We thus have a certain correspondence between the bundle of directions through O and the points of p'. For directions infinitesimally close to the direction of c at O, and for points close to O', this correspondence is linear; and by a proper selection of cartesian axes at O and O', we may write the correspondence in the canonical form

$$\xi = \alpha_1 x', \quad \eta = \beta_1 y',$$

where (x', y') denote the coordinates of the point in the plane p', and the corresponding direction at O has direction cosines proportional to $(\xi : \eta : 1)$.

In an entirely analogous way, by considering the curves of the natural family which go through O', and the points of intersection with the plane p, we obtain a second linear correspondence which may be reduced to the form

$$\xi' = \alpha_2 x, \quad \eta' = \beta_2 y,$$

where (x, y) is the point in the plane p and $(\xi' : \eta' : 1)$ gives the corresponding direction at O'.

If we were dealing with an arbitrary family of ∞^4 curves, instead of a natural family, these linear correspondences would still exist; but the choice of axes in the second canonical form would be different from that required in the first, and the two constants appearing in the second form would be independent of those

appearing in the first. *The peculiarity of thè natural type may be stated in the following form: First, the canonical axes for the two correspondences coincide; second, the ratio of the characteristic constants has the same value for both correspondences.*

This is the essential geometric content of the long statement quoted above from Thomson and Tait. Is this characteristic of the natural type? We do not know.

64′. A statement in more concrete terms is of interest. If we start out from O in directions equally inclined (the fixed angle is of course assumed infinitesimal) to the direction of c, that is, along a cone of revolution having for axis the tangent of c, the resulting trajectories forming a sort of curvilinear cone, we strike points on p' located on an ellipse with O' as center. By changing the angle of the cone we obtain a family of similar and similarly situated ellipses. The principal axes of these ellipses are the canonical directions referred to above for the first correspondence, and the ratio of the diameters is equal to the ratio of the canonical constants $(\alpha_1 : \beta_1)$. By starting from the other point O' along cones of revolution having for axis the tangent to c, we strike the plane p in a second set of homothetic ellipses. *The two sets of ellipses thus obtained, one in the plane p, and the other in the plane p', are similar.* This is part of the property stated, but not the whole. It should be observed that it has no meaning to say that the two sets are similarly situated, since they are in different planes.

65. We may, however, obtain two sets in the same plane as follows: If we start along the cone of revolution from O, we hit p' in an ellipse. If we wish to hit p in a circle, we must start at O' along a certain elliptical cone: the sections of this cone by planes parallel to p', projected orthogonally on p', give a set of homothetic ellipses. We thus have in the plane p', two sets of ellipses, the first set being obtained from cones of revolution at O, and the second set being obtained from elliptical cones at O' by orthogonal projection of parallel sections. If we were dealing with an arbitrary family of curves, the two sets thus obtained would be unrelated: *for a natural family, however, the two sets coincide.*

66. Of course we could also construct two sets in the plane p and these would coincide; but this would not give an additional property. In the statement quoted, certain pairs of congruent instead of merely similar ellipses appear, but that is due to the introduction of kinematics: namely, use is made of the velocities V and V' at the points O and O'. "If O and O' are regarded as optic centers of the eyes of two persons looking at one another through any optical apparatus, and if their pupils are of equal size in reality, they will be seen as similar ellipses of equal apparent dimensions by the two observers." It should be observed, however, that the dimensions will be *equal* only under the assumption that the two eyes are at positions for which the velocities V and V', or, what is equivalent, the indices of refraction ν and ν', are equal. In the most general case of an isotropic medium, the ellipses will not have equal apparent dimensions, but the ratio of the dimensions will be equal to the ratio of the two velocities.

67. Two converse questions remain unanswered. First: Find all systems of ∞^4 curves in space such that circles about O and O' appear as similar ellipses.

Second: Find all systems such that the set of ellipses in the plane p' formed by starting from O along cones of revolution, and the set of ellipses found by orthogonal projection upon p' of the sections cut out by planes parallel to p' of those (elliptical curvilinear) cones at O' which strike plane p in circles,—such that these two sets of ellipses shall coincide.

§§ 68–72. The So-called General Problem of Dynamics

68. Consider any material system (particles or rigid bodies) with n degrees of freedom, so that its position at each instant is determined by n independent coordinates denoted by $x_1, x_2, \cdots, x_n$. The kinetic energy T will be represented by a quadratic form

$$2T = \sum a_{ik}\dot{x}_i\dot{x}_k,$$

where the coefficients a are functions of the coordinates, and

the dots denote time derivatives. If the acting forces are conservative, there will exist a force function $W(x_1, x_2, \cdots, x_n)$, which is assumed to be independent of the time, and the equation of energy

$$T - W = h$$

asserts that in any given motion the sum of the kinetic and potential energies is constant.

The so-called general problem of dynamics requires the determination of the motions when we are given the form T, the function W, and the constant h. The possible trajectories are then given by the Jacobi principle of least action as the extremals of the integral

$$\int \sqrt{W + h}\sqrt{\Sigma a_{ik}dx_idx_k}.$$

This defines the *most general natural family*. The integral is of the form $\int Fds$, where F is any point function and ds is the length-element in a general n-dimensional variety V_n defined by

$$ds^2 = \Sigma a_{ik}dx_idx_k.$$

69. Such a family consists of $\infty^{2(n-1)}$ curves, in the space V_n, one passing through each point in each direction. A *complete characterization* is given by J. Lipke, in his doctor's dissertation,* as follows:

(A_1) The locus of the centers of geodesic curvature of the ∞^{n-1} curves passing through any point of V_n is a flat space of $n - 1$ dimensions S_{n-1}.

(A_2) The osculating geodesic surfaces (two-dimensional varieties) at the given point form a bundle of surfaces, all containing a fixed direction (and hence the geodesic line in that direction) which is normal to the S_{n-1} of property A_1.

(B) The n directions at any point, in which, as a consequence of the preceding properties, the osculating geodesic circles (circles

* *Trans. Amer. Math. Soc.*, vol. 13 (1912), pp. 77–95.

of constant geodesic curvature) hyperosculate the curves of the given family, are mutually orthogonal.

70. This gives the generalization of properties A and B stated in §§ 29–31. The simpler results there given for ordinary space apply to a euclidean space of any dimensionality and also to spaces of constant curvature. In the general space of variable curvature, the geodesic circles constructed at a given point do not all meet at a second point, and so no analogue of the law of reciprocity of natural families presents itself.

71. The theorem of Thomson and Tait remains valid for any space.* The converse questions connected with it have not been settled. In all probability the Thomson-Tait geometric property is characteristic in any space (flat or curved) of dimensionality greater than two. Obviously in the case of two dimensions the geometric converse is not valid, since any system of ∞^1 curves admits ∞^1 orthogonal curves.

72. The systems characterized by property A (meaning A_1 together with A_2) are the most general velocity systems in V_n. The case $n = 2$ presents a peculiar feature: for then, included in the velocity type, we have, in addition to the natural type, another special type of interest (geometric, rather than dynamic), namely the isogonal type† (systems formed by the ∞^2 isogonal trajectories of an arbitrary simply infinite system of curves). In the case of the plane (or any surface of constant curvature) the reciprocity construction for velocity systems is available, and each of the species, natural and isogonal, is self-reciprocal. The only families common to the two species are those formed by the isogonals of an isothermal system, or, what is the same, by velocity systems generated by Laplacian fields of force.‡

* Cf. Darboux, Leçons, vol. 2, last chapter, where references to the memoirs of Lipschitz and Beltrami are given.

† Scheffers introduced the systems of plane curves $y'' = (\psi - y'\varphi)(1 + y'^2)$ in connection with the theory of isogonals, and obtained a law of reciprocity for isogonal systems. Cf. *Leipziger Berichte*, 1898, 1900; *Mathematische Annalen*, vol. 60.

‡ Cf. the author's note, "Isothermal systems in dynamics," *Bull. Amer. Math. Soc.*, vol. 14 (1908), pp. 169–172.

We note finally this characteristic distinction between the two noteworthy species:

For both natural and isogonal families in the plane, the circles of curvature constructed at any point p have another point P in common. The point transformation T (from p to P) in the natural case is such that the two lineal elements at any point, each of which is converted into a cocircular element, are orthogonal; while in the isogonal case the two elements, each of which is converted into an element normal to a cocircular element, are orthogonal.

If the transformation T connected with a velocity system is required to be (direct) conformal, the corresponding field must be Laplacian. Such fields are distinguished from all others by the fact that each of the infinitude of systems of velocity curves is then expressible linearly in the two parameters involved.

CHAPTER III

TRANSFORMATION THEORIES IN DYNAMICS

§§ 73–81. PROJECTIVE TRANSFORMATIONS

73. The general object of a transformation theory is to relate new problems to old problems, and so to proceed from the solution of the latter to the solution of the former. The most important geometric transformations are the projective and the conformal. Both groups play important rôles in dynamics, the former in connection with general fields, and the latter in connection with conservative fields.

74. The importance of projective transformations in dynamics was brought out by Appell in 1889. Given any positional field of force in the plane, the corresponding equations of motion are of the form

$$(1) \qquad \frac{d^2x}{dt^2} = \varphi(x, y), \qquad \frac{d^2y}{dt^2} = \psi(x, y).$$

If an arbitrary point transformation, unaccompanied by any change in the time, is applied, the new differential equations will usually involve not only x and y, but also the velocity components dx/dt, dy/dt. In fact the only exception is where the point transformation is merely affine:

$$x_1 = ax + by + c, \quad y_1 = a'x + b'y + c'.$$

Appell showed that if a general collineation

$$(2) \qquad x_1 = \frac{ax + by + c}{a''x + b''y + c''}, \qquad y_1 = \frac{a'x + b'y + c'}{a''x + b''y + c''}$$

is accompanied by a change of the time of the form

$$(2') \qquad dt_1 = \frac{dt}{k(a''x + b''y + c'')^2},$$

the new differential equations will be of the original form

$$(3) \qquad \frac{d^2x_1}{dt_1^2} = \varphi_1(x_1, y_1), \quad \frac{d^2y_1}{dt_1^2} = \psi_1(x_1, y_1),$$

and therefore define motion in some new positional field of force. The relation between the new field and the original field is explicitly as follows

$$(4) \qquad \begin{aligned} \varphi_1 &= k^2(a''x + b''y + c'')^2\{C'(x\psi - y\varphi) + B'\varphi - A'\psi\}, \\ \psi_1 &= k^2(a''x + b''y + c'')^2\{-C(x\psi - y\varphi) - B\varphi + A\psi\}, \end{aligned}$$

where the capital letters denote minors in the determinant $|ab'c''|$ of (2).

74′. The trajectories of the original field are converted by the collineation into the trajectories of the new field. Also the directions of forces of the two fields are projectively related. It must not be thought, however, that the force vector acting at a given point (x, y) in the first plane is projected into the new force vector acting at the point (x_1, y_1) of the second plane: the initial points of the two vectors will correspond, of course, by the given collineation, but the terminal points will not. The question therefore arises, *what is the geometric relation between the new vector field and the old vector field?*

To answer this question we take our rectangular axes so that the collineation takes its metrical normal form. (Affinities of course require a separate discussion.) The canonical formulas for our transformation are

$$(5) \qquad \begin{gathered} x_1 = \frac{\gamma\gamma_1}{x}, \quad y_1 = \frac{\gamma_1 y}{x}, \\ \varphi_1 = -k^2\gamma\gamma_1 x^3\varphi, \quad \psi_1 = k^2\gamma_1 x^2(x\psi - y\varphi), \end{gathered}$$

together with

$$(5') \qquad dt_1 = \frac{dt}{kx^2}.$$

To each collineation between the two planes corresponds a defi-

nite vector transformation. The vectors are here of the third type (*bound* vectors) described in the Introduction, requiring four coordinates for their determination. The original vector is defined by the four numbers (x, y, φ, ψ), the first two defining the initial point, and the last two giving the components of the vector. The coordinates of the new vector are $(x_1, y_1, \varphi_1, \psi_1)$.

The vector transformation induced by the given collineation is not projective. The new vector has the same initial point and the same direction as the projection of the old vector, but has a different length. The ratio λ between the actual length of the new vector and the length of the projected vector is

$$(5'') \qquad \lambda = k^2x^3(x + \varphi).$$

Noting that in the canonical form x and $x + \varphi$ denote the distances from the initial and terminal points of the original vector to the vanishing line in the first plane, we may state this result.

Any given (non-affine) collineation* (2) *induces a certain vector transformation (determined up to the factor k) defined analytically by* (2) *and* (4), *and geometrically as follows: If PQ is any bound vector in the first plane, and if the collineation converts the initial point P into P_1 and the terminal point Q into Q_1, then the transformed bound vector is not P_1Q_1, but P_1Q_1', where Q_1' is the point on the line joining P_1Q_1 such that the ratio $\lambda = P_1Q_1'/P_1Q_1$ equals k^2 times the cube of the distance from P to the vanishing line times the distance from Q to that vanishing line.*

The transformation converts the ∞^4 bound vectors of the first plane, represented by the independent coordinates (x, y, φ, ψ), into the ∞^4 bound vectors of the new plane.† In the dynamical application, φ and ψ are given as functions of x, y, that is, we have a field of ∞^2 vectors, one for each initial point: the result of

* In the case of an affine collineation, the induced vector transformation is, except for the constant factor k, merely the result of applying the affinity to both ends of the vector. It is thus linear.

† The vector transformations induced by inverse collineations are inverse to each other. The four-dimensional transformations are therefore Cremona transformations.

the transformation is a new field, φ_1 and ψ_1 being expressible in terms of x_1, y_1. The ∞^3 trajectories of the first field are converted by the collineations into the ∞^3 trajectories of the new field; it is to be noticed however that, during any corresponding motions, positions which correspond according to the collineation will usually not correspond to the same instant of time; in fact from (2′)

$$t_1 = \int \frac{dt}{k^2(a''x + b''y + c'')^2}.$$

75. If X, Y denote the velocity components at the position x, y and if the corresponding velocity in the second plane is X_1, Y_1, acting at the position x_1, y_1, then we find, from the canonical form (5),

$$\begin{aligned} x_1 &= \gamma\gamma_1/x, & y_1 &= \gamma_1 y/x, \\ X_1 &= -k\gamma\gamma_1 X, & Y_1 &= k\gamma_1(xY - yX). \end{aligned} \tag{6}$$

Thus we have a different vector transformation which may be termed the *phase* transformation* (in distinction from the *force transformation* of § 74): it gives the relation between the corresponding phases in the two planes.

If we speak of points and vectors which correspond in the two planes according to the given collineation as projectively related, then the result may be stated in this form:

The new phase vector does not coincide with the projection of the given phase vector: it has the same initial point, but the ratio of the actual length to the length of the projected vector is k^2 times the product of the distances from the ends of the original vector to the vanishing line of the collineation.

76. Having studied the Appell transformation and its geometric interpretation in terms of force vectors and phase vectors, we now ask whether other more general transformations can play a like rôle. Appell proved the following converse theorem:

* The phase of a particle at any instant, in the sense of Gibbs, is its position together with its velocity: it is defined by the four numbers $(x, y, \dot{x}, \dot{y})$.

The only transformations of the form

$$x_1 = \Phi(x, y), \qquad y_1 = \Psi(x, y), \qquad dt_1 = \mu(x, y)dt$$

which convert every set of differential equations

$$(1) \qquad \frac{d^2x}{dt^2} = \varphi(x, y), \qquad \frac{d^2y}{dt^2} = \psi(x,y),$$

into one of the same form are those defined by (2), (2′).

77. By eliminating the time from (1), giving the differential equation of the trajectories in the form (page 7)

$$(7) \quad (\psi - y'\varphi)y''' = \{\psi_x + (\psi_y - \varphi_x)y' - \varphi_y y'^2\}y'' - 3\varphi y''^2,$$

the author proved that the only point transformations which convert every trajectory system (of a positional field) into a trajectory system are the collineations. This remains valid even in the domain of all contact transformations, as we now proceed to show.

We first consider the class of differential equations (cf. page 11)

$$(8) \qquad y''' = G(x, y, y')y'' + H(x, y, y')y''^2$$

including (7) as a special case, and characterized geometrically by the possession of property I (that is, the focal locus for each element is a circle through the given point). We prove this theorem:

The only contact transformations which convert every equation of type (8) (*that is, every system of curves with property I*) *into one of the same type are collineations and correlations.*

That no other transformations are possible is seen as follows. If a contact transformation is to convert type (8) into itself, it must convert the part common to all systems of that type into itself. The curves defined by $y'' = 0$, that is, straight lines, obviously satisfy (8) for every form of G and H. It is obvious that no other (proper) curves satisfy all such equations. But since we are dealing with contact transformations and not merely point transformations, we must replace the concept *curve* by

the concept *union.* In the plane the only unions which are not (proper) curves are points. A point is regarded as made up of ∞^1 lineal elements; so x is constant, y is constant, y' is arbitrary, and therefore y'' and y''' are infinite. Point unions are to be regarded then as solutions of all equations (8). The common part thus consists of the ∞^2 straight lines and the ∞^2 points of the plane. If this is to go into itself, either points go into points and lines into lines, or else points go into lines and lines into points. We thus obtain only collineations and correlations.

That the collineations actually leave type (8) unchanged is easily verified analytically.* The work for correlations is simplified by observing that every correlation may be reduced, by means of collineations, to the form of Legendre's transformation

$$(9) \qquad x_1 = -y', \qquad y_1 = xy' - y, \qquad y_1' = -x,$$

(which is simply polarity with respect to the conic $x^2+2y-1=0$). Extending (9), we find

$$(9') \qquad y_1'' = \frac{1}{y''}, \qquad y_1''' = \frac{y'''}{y''^2}.$$

This converts equation (8) into one of the same form

$$(10) \qquad y_1''' = G_1(x_1, y_1, y_1')y_1'' + H_1(x_1, y_1, y_1')y_1''^2,$$

the new coefficient functions being related to the old as follows:

$$(10') \qquad \begin{aligned} G_1 &= H(-y_1', x_1y_1' - y_1, -x_1), \\ H_1 &= G(-y_1', x_1y_1' - y_1, -x_1). \end{aligned}$$

This completes the proof of the theorem stated on the previous page.

78. If we impose property II on the system (8), that is, if we consider the subclass in which

$$(11) \qquad H = \frac{3}{y' - \omega(x, y)},$$

* *Trans. Amer. Math. Soc.*, vol. 7 (1906), p. 420.

the correlations are no longer available. That collineations actually convert this subclass into itself is readily verified. The same is true for the still narrower class, characterized by properties I, II, and III, in which the differential equation is of the form (cf. page 13)

$$(12) \qquad (y' - \omega)y''' = \{\lambda y'^2 + \mu y' + \nu\}y'' + 3y''^2.$$

79. We pass now to the case of dynamical trajectories, defined by type (7), and state the fundamental result:

Collineations are the only contact transformations of the plane which convert every system of ∞^3 dynamical trajectories (belonging to an arbitrary positional field of force) into such a system.

The only possibilities here also are collineations and correlations. The former actually have the required property. The latter have not, as is seen by observing that the application of the Legendre transformation (9) to a dynamical equation (8) will result in a new equation, which, while still of the general form (8), will not usually be of the dynamical form.*

80. Systems of trajectories are characterized by the set of five geometric properties of page 10. Therefore projective transformation will convert any system of curves having these properties into a system having the same properties. So, in spite of the fact that the properties as stated involve metric ideas (osculating parabolas, angles, circles of curvature, etc.), the set is actually projectively invariant. It ought to be possible therefore to *restate the geometric characterization in projective language.*

We shall not attempt to carry out this idea completely, and merely restate properties I and II as follows:

Consider the ∞^1 trajectories passing through a given point O in a given direction whose slope is y'. For each of these trajectories construct the conic which has four-point contact at O and touches the line determined by two arbitrarily selected

* We see from (10′) that the coefficients G and H, which are rational with respect to y', are converted into coefficients which are not usually rational.

points* A and B (which remain fixed in the following statements); through A and B draw tangents to the conic (in addition to the fixed line) and join the points of contact. *The lines thus constructed, one for each of the* ∞^1 *trajectories, will form a pencil* (property I).

As the initial direction (*that is* y') *varies about* O, *the vertex of the pencil just described will move along a straight line*† *passing through* O (property II).

The other properties, especially the fifth, are much more complicated.

81. In conclusion we point out another way in which the projective group enters in dynamics. If an arbitrary point transformation

$$x_1 = \Phi(x, y), \qquad y_1 = \Psi(x, y)$$

is applied to the differential equations

$$\ddot{x} = \varphi(x, y), \qquad \ddot{y} = \psi(x, y),$$

defining motion under a purely positional force, the new differential equations, of the more general form

$$\Phi_x\ddot{x} + \Phi_y\ddot{y} + \Phi_{xx}\dot{x}^2 + 2\Phi_{xy}\dot{x}\dot{y} + \Phi_{yy}\dot{y}^2 = \varphi(\Phi, \Psi),$$

$$\Psi_x\ddot{x} + \Psi_y\ddot{y} + \Psi_{yy}\dot{y}^2 + 2\Psi_{xy}\dot{x}\dot{y} + \Psi_{yy}\dot{y}^2 = \psi(\Phi, \Psi),$$

will usually define a motion due to a positional force together with a force depending on the velocity $\dot{x}$, $\dot{y}$. If this latter force is to be absent the transformation will be affine, as already remarked (§ 74). If, instead, we demand that the latter force shall act in the direction of the velocity (and thus be in the nature of a resistance), we find that the transformation may be any collineation.

More generally, *projective transformations are the only point*

* In the original metric statements these are of course the circular points at infinity.

† The force direction will be determined projectively as the harmonic of this line with respect to the lines joining O to A and B.

transformations which leave invariant the type

$$\ddot{x} = \varphi(x, y) + \dot{x}R(x, y, \dot{x}, \dot{y}),$$
$$\ddot{y} = \psi(x, y) + \dot{y}R(x, y, \dot{x}, \dot{y}),$$

defining motion of a particle under any positional force together with any resistance term acting in the direction of motion.

§§ 82–91. CONFORMAL TRANSFORMATIONS

82. The importance of conformal transformation is well known in connection with the theory of the potential. Geometric inversion or transformation by reciprocal radii, for example, yields the method of electric images due to Sir William Thomson. In connection with dynamics, the importance of general conformal transformations has been emphasized by Larmor, Goursat, and Darboux.*

83. Consider any conformal representation of the points of two surfaces S and S_1. The first fundamental forms of the surfaces may be taken to be

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$
$$ds_1{}^2 = \lambda(Edu^2 + 2Fdudv + Gdv^2),$$

where corresponding points have the same parameters u, v. *The principal theorem is that every natural system on one surface becomes by the conformal representation a natural system on the other.* This is obvious if we remember that natural systems are obtained by minimizing an integral in which the integrand is the element of length multiplied by any point function. Hence

The only point transformations (in any space) which convert every natural family into a natural family are the conformal.

84. Consider now the ∞^3 dynamical trajectories on S produced by a conservative field of force, the work function being W. These consist of ∞^1 natural families, one for each value of the

* Cf. the discussion in Routh, Dynamics of a Particle, Nos. 628–635 (method of inversion and conjugate functions).

constant of total energy h. It will be convenient to refer to the particular natural system produced in the given field W for a particular value h, as the family due to $W + h$.

The corresponding family on S_1 is due to

$$\frac{W + h}{\lambda}.$$

Hence the ∞^1 related natural families on S, found by varying h, go over by the conformal representation into ∞^1 natural families which are *not* usually related, that is, do not form the complete system of trajectories belonging to a conservative field. The only case in which the new families are related arises when

$$W = \lambda,$$

for then the new systems are due to the work function

$$W_1 = 1/\lambda.$$

We then reach the conclusion that *in any conformal representation (excluding the trivial homothetic case*) there is a unique conservative force whose complete system of ∞^3 dynamical trajectories is converted into the complete system of some (usually distinct) conservative force. The work function of the force in question is defined by the squared ratio of magnification,*

$$W = \lambda = \frac{ds_1^2}{ds^2}.$$

85. Similar statements may be made for brachistochrones. Every system of ∞^2 brachistochrones due to any work function and a given value of h of course becomes such a system, for any natural family may be regarded as a family of brachistochrones. But *there is only one complete system of ∞^3 brachistochrones which is converted into a complete system, namely, that defined by the work*

* It is obvious that in this case *every* complete system of trajectories becomes a complete system. The same holds for brachistochrones and catenaries.

function

$$W = 1/\lambda.$$

For any other work function the ∞^1 families of brachistochrones, due to $W + h$, become ∞^1 non-related natural families on S_1 due to

$$\lambda(W + h).$$

86. In the case of catenaries due to $W + h$, the ∞^1 usually non-related natural families corresponding on S_1 are due to

$$\frac{W + h}{\sqrt{\lambda}}.$$

Hence *the only complete system of catenaries which is turned into a complete system is defined by the work function**

$$W = \sqrt{\lambda}.$$

87. Consider, for example, the conformal representation of the plane

$$z = x + iy = re^{i\theta}$$

on the plane

$$z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$$

defined by

$$z_1 = z^n,$$

where n is neither 0 nor 1.

Here the squared ratio of magnification is

$$\lambda = r^{2(n-1)} = r_1^{\frac{2(n-1)}{n}}$$

* The three physical cases mentioned may be included in one general discussion by considering the extremals of

$$\int v^m ds \equiv \int (W + h)^{m/2} ds = \text{minimum};$$

when $m = 1$, we have least action and trajectories; when $m = -1$, least time and brachistochrones. For every value of m we obtain, by varying h, a system of ∞^3 curves. Cf. the general discussion of the systems S_k defined (for arbitrary fields) in Chapter IV.

Applying the theorems stated above, we find that the trajectories generated by

$$W = r^{2(n-1)}$$

go over into the trajectories of a new field

$$W_1 = r_1^{2\left(\frac{1}{n}-1\right)}.$$

For brachistochrones the corresponding fields are

$$W = r^{-2(n-1)}, \qquad W_1 = r_1^{-2\left(\frac{1}{n}-1\right)};$$

and for catenaries

$$W = r^{n-1}, \qquad W_1 = r_1^{\frac{1}{n}-1}.$$

The particular transformation $z_1 = z^2$, that is, $n = 2$, gives rise to simple fields. Stating the results in terms of the law of the central forces obtained, instead of the corresponding work functions, we have:

The trajectories of a central force varying as r (that is, the conics described about the center of force as center) become the trajectories of a central force varying as r_1^{-2} (that is, the conics described about the center of force as focus).

The brachistochrones of a central force varying as r^{-3} become the brachistochrones of a central force of constant intensity.

The catenaries of a central force of constant intensity become the catenaries of a central force varying as $r_1^{-3/2}$.

88. Returning to the general conformal representation, we observe that ∞^1 natural families forming a complete system of trajectories can never become a complete system of brachistochrones. For the trajectories on S due to $W + h$ become ∞^1 natural families on S_1, which, when regarded as brachistochrones, are due to $\lambda/(W + h)$; and there is no work function which reduces this expression to the form of a function of u, v plus a constant depending only on h. Thus for a given (non-homothetic) conformal transformation there is *one* system of trajectories

which is converted into a system of trajectories, and *one* system of brachistochrones which is converted into a system of brachistochrones, but there is *no* system of trajectories which is converted into a system of brachistochrones. The same is true for any two of the three types trajectories, brachistochrones, catenaries or of the infinite number of types described in the preceding footnote (page 83).

89. As another application, consider the *velocity curves* connected with a plane field of force whose work function is $W(x, y)$. For a given speed v_0, we obtain ∞^2 such curves, defined by the property that the curvature at each point and direction equals the curvature of a free particle starting out from that point and direction with the speed v_0. The differential equation of this velocity system is

$$y'' = \frac{(W_y - y'W_x)(1 + y'^2)}{v_0^2}.$$

This is recognized as a natural family; it corresponds to the geodesics of the surface whose first fundamental form is

$$e^{\frac{2W}{v_0^2}}(dx^2 + dy^2).$$

By varying v_0 we obtain the ∞^1 velocity systems belonging to the given field; they are pictured by the geodesics of ∞^1 surfaces.

Consider now a conformal representation of the xy-plane upon itself. This converts $dx^2 + dy^2$ into

$$e^H(dx^2 + dy^2),$$

where $H(x, y)$, by known theory, is a harmonic function. We thus obtain ∞^1 new natural families corresponding to the geodesics of the ∞^1 surfaces

$$e^{\frac{2}{v_0^2}W + H}(dx^2 + dy^2).$$

These ∞^1 natural families cannot usually be regarded as related velocity systems for some new field: the requisite condition is that W shall be the same as H except for a constant factor.

Hence for a given conformal transformation of the plane (which is not merely a similitude), there is a unique complete velocity system belonging to a conservative field of force which is converted into a complete system. The unique work function is

$$W = H = \log \lambda,$$

where λ denotes the squared ratio of magnifaction in the given conformal representation. The fields obtained are *Laplacian*, that is, satisfy the condition

$$W_{xx} + W_{yy} = 0.$$

As an example, the transformation $z_1 = \log z$ converts the ∞^3 velocity curves of the field $W = \log r$ (in which the force varies inversely as the distance from the origin) into the ∞^3 velocity curves of the field $W_1 = x_1$ (force vertical and constant).

90. It was shown above that conformal transformations are the only point transformations which convert every natural family into a natural family. Natural families are characterized by properties A and B of § 31. It is of interest to notice that property A by itself is conformally invariant. The most general system having this property (that osculating circles constructed at any point have another point in common) is what we have termed a velocity system. We now prove that

The only point transformations which convert every velocity system into a velocity system are the conformal transformations.

Consider say the three-dimensional case, where the general velocity system is

$$y'' = (\psi - y'\varphi)(1 + y'^2 + z'^2), \qquad z'' = (\chi - z'\varphi)(1 + y'^2 + z'^2).$$

The only curves which are common to all such systems must

satis y

$$1 + y'^2 + z'^2 = 0, \qquad y'' = 0, \qquad z'' = 0,$$

and are therefore the minimal straight lines of space. Since the only transformations converting minimal lines into minimal lines are conformal, we have the result stated. That conformal transformations actually leave the velocity type invariant is easily verified analytically*. The result is obvious synthetically (in the case of more than two dimensions) since the conformal group converts circles into circles and bundles of circles into bundles. Hence if the original system possesses property A, the same will be true of the transformed system.

91. It may be shown that, for any given non-conformal transformation, there exists one and only one velocity system which is converted into a velocity system.

§§ 92–94. Contact Transformations

92. With each natural family, or, what is the same, with each isotropic medium, there is associated a definite infinitesimal contact transformation. This connection, which appears implicitly in Hamilton's fundamental memoir of 1835, was worked out in detail by S. Lie.†

If the index of refraction is $\nu(x, y, z)$, the associated contact transformation has the characteristic function

$$\nu(x, y, z) \sqrt{1 + p^2 + q^2}, \tag{1}$$

where x, y, z, p, q are considered as the coordinates of a surface element. If the one-parameter group generated is applied to an arbitrary surface the resulting ∞^1 surfaces form a wave set. The trajectories or rays appear as the path curves of this group. Lie showed that the category of transformations which thus

* Cf. *American Journal of Mathematics*, vol. 27 (1906), p. 213, for the two-dimensional case.

† "Die infinitesimalen Berührungstransformationen der Mechanik," *Leipziger Berichte* (1889), pp. 145–153. A very elegant discussion, with new results, is given by Vessiot, *Bull. Soc. math. de France*, vol. 34 (1906), pp. 230–269.

appears, with a characteristic function of type (1), and which he termed "the infinitesimal contact transformations of mechanics," is distinguished geometrically by the fact that the so-called* *transversality* relation reduces to orthogonality.

93. The following simple and easily proved theorem appears to be new.

The alternant (or Klammerausdruck of Lie) of the contact transformations associated with any two media is always a point transformation.

94. Here we are dealing with two natural families in the same three-dimensional space. In connection with the most general problem of dynamics (page 70), spaces of any dimensionality must be considered, with arbitrary variable curvature. The space depends on the quadratic form defining the kinetic energy: this determines the quadratic expression appearing under the radical in the generalization of (1). The potential† determines the factor ν which may be any point function. The general theorem is then as follows:

The alternant of the contact transformations associated with two dynamical problems (or natural families) will be a point transformation when, and only when, the two expressions for the kinetic energy are either the same or differ by a factor (which may be any point function); the two potential energies remain entirely arbitrary.

In particular, if any two natural families are constructed in the same space (which space is entirely arbitrary), the alternant will be a point transformation.

For a detailed discussion of the two-dimensional case, including a number of converse results, the reader is referred to the author's paper, cited in the first footnote below.

* Lie does not use this term. The author borrows it from the closely connected problem in the calculus of variation. See "The infinitesimal contact transformations of mechanics," *Bull. Amer. Math. Soc.*, vol. 16 (1910), pp. 408–412.

† Here considered as including the energy constant h, which is fixed, since we are dealing with a natural family.

§§ 95–97. A Group of Space-time Transformations

95. In the fundamental transformation of the relativity theory, known as the Lorentz transformation, the position coordinates x, y, z and the time coordinate t are merged: the new position and the new time appear as functions of both the original position and the original time. The Lorentz group is composed of the linear transformations of the four variables x, y, z, t which leave invariant the quadric

$$x^2 + y^2 + z^2 - c^2t^2 = 0.$$

Its importance is due to the fact that it leaves unaltered the form of the Maxwell equations.

We consider in this section an entirely different group of space-time transformations, depending on arbitrary functions instead of arbitrary constants. It arises in connection with ordinary (newtonian) dynamics in the theory of forces depending on the time as well as position.

We confine the discussion for the sake of simplicity to the case of two dimensions. What transformations of the three variables x, y, t will convert any set of equations of the form

$$(1) \qquad \frac{d^2x}{dt^2} = \varphi(x, y, t), \qquad \frac{d^2y}{dt^2} = \psi(x, y, t)$$

into another set of the same form? An arbitrary transformation would produce equations representing a force depending, not only on x, y, t, but also on the velocity dx/dt, dy/dt. The problem is to find those peculiar transformations which do not introduce the velocity in the final equations. The result is as follows:

The only space-time transformations which convert every space-time field of force into a space-time field are those of the form

$$(2) \qquad t_1 = f(t), \qquad x_1 = (ax + by)\sqrt{f'(t)} + g(t),$$
$$y_1 = (cx + dy)\sqrt{f'(t)} + h(t).$$

The group thus involves three arbitrary functions $f(t)$, $g(t)$, $h(t)$ as well as four arbitrary constants a, b, c, d.

96. Another representation of the same group, which has the advantage of avoiding radicals, is

$$(3)\qquad \frac{dt_1}{dt} = \{\lambda(t)\}^2, \qquad x_1 = (ax + by)\lambda(t) + \mu(t),$$
$$y_1 = (cx + dt)\lambda(t) + \nu(t).$$

When such a transformation is applied to equations (1), the new equations are found to be

$$\lambda^5\ddot{x}_1 = (\lambda\ddot{\lambda} - 2\dot{\lambda}^2)(ax + by) + \lambda^2(a\varphi + b\psi) + \lambda\ddot{\mu} - 2\dot{\lambda}\dot{\mu},$$
$$\lambda^5\ddot{y}_1 = (\lambda\ddot{\lambda} - 2\dot{\lambda}^2)(cx + dy) + \lambda^2(c\varphi + d\psi) + \lambda\ddot{\nu} - 2\dot{\lambda}\dot{\nu}.$$

Of course the original variables x, y, t are here to be replaced by their values in the new variables x_1, y_1, t_1.

97. The transformation converts the space-time curves of the original force into the space-time curves of a new force. Of course it is not a point transformation of the xy-plane, so it does not, as was the case for the Appell transformation (page 76), convert trajectories into trajectories. These remarks apply even in the special case where the force is positional. Consider, as a simple example, the transformation

$$t_1 = \tfrac{1}{2}e^{2t}, \qquad x_1 = xe^t, \quad y_1 = ye^t,$$

applied to the equations

$$\ddot{x} = x, \qquad \ddot{y} = y.$$

The transformed equations are found to be

$$\ddot{x}_1 = 0, \qquad \ddot{y}_1 = 0.$$

The first field is central, the force varying directly as the distance, so that the trajectories are ∞^3 conics with the same center. The second force is everywhere zero, so the trajectories are merely ∞^2 straight lines.

CHAPTER IV

CONSTRAINED MOTIONS IN A FIELD. GENERALIZATION OF THE TRAJECTORY PROBLEM INCLUDING BRACHISTOCHRONES AND CATENARIES

§§ 98–114. Systems S_k Defined by $P = kN$

98. In connection with a field of force, the only curves usually studied are the lines of force and the trajectories. In the plane the lines of force form a simply infinite system, and the trajectories a triply infinite system. The former system has no peculiar properties, since any set of ∞^1 curves may be regarded as the lines of force in some field, in fact in an infinite number of different fields. The triply infinite system of trajectories has peculiar properties which have been discussed in Chapter I. Other noteworthy systems of curves are connected with the field, for example, brachistrochrones, catenaries. velocity curves, and tautochrones.

99. Omitting the tautochrones, *the other three systems named, together with the trajectories, may all be obtained as special cases of this simple general problem*: to find curves along which a constrained motion is possible such that the pressure is proportional to the normal component of the force.

100. If an arbitrary curve is drawn in the plane field of force, and the particle, of say unit mass, is started along it from one of its points with a given speed, the constrained motion along the given curve is determined. The acceleration along the curve is given by T, the tangential component of the force vector. So the speed at any point is determined by

$$v^2 = \int T ds. \tag{1}$$

The pressure P (of course normal to the curve, since the curve

is considered smooth) is given by the elementary formula

$$P = \frac{v^2}{r} - N. \tag{2}$$

If we increase the initial speed, the effect is to increase v^2 by a constant c; and hence P changes by the addition of a term of the form c/r.

101. If the given curve is a trajectory, the initial speed may be so chosen that the pressure vanishes throughout the motion; that is, trajectories may be defined as curves of no constraint. Of course, if a different initial speed is used, P will be of the form c/r; but, as regards the curves, they are completely characterized by $P = 0$.

102. If the given curve is a brachistochrone and if the motion along it is brachistochronous, Euler proved (assuming the force to be conservative) that the pressure was double the normal component of the acting force and opposite to it in direction, that is, $P = -2N$. If the force is not conservative, the real brachistochrones, as defined by a problem of the calculus of variations, form a quadruply infinite system. The curves defined by the property $P = -2N$ then form a triply infinite system of what should be called pseudo-brachistochrones. These curves are really brachistochrones only in the conservative case. No ambiguity however will arise by terming the system here considered brachistochrones instead of pseudo-brachistochrones.

103. *The general problem suggested is to find curves such that P shall be proportional to N.* So $P = kN$. To a given value of k there correspond ∞^3 such curves: the system so obtained will be denoted by S_k. *The four special cases of physical interest are as follows:*

$k = 0$ gives S_0, the system of *trajectories*;

$k = -2$ gives S_{-2}, the system of *brachistochrones*;

$k = 1$ gives S_1, the system of *catenaries*;

$k = \infty$ gives S_∞, the system of *velocity curves.*

104. The last case requires a justification in terms of limits which is easily carried out analytically.

105. The third case follows from the known fact that when an inextensible flexible homogeneous string is suspended in any field of force, the resulting form of equilibrium, called a catenary in the general sense of the term, has the dynamical property that when a particle, started out with the proper initial velocity, rolls along the curve, the pressure at any point equals the normal component of the force: that is, catenaries are defined by $P = N$, corresponding to $k = 1$.

106. Of course a triply infinite system S_k exists for any value of the parameter k. The differential equation of the system, in intrinsic form, is easily obtained by eliminating v from the equations

$$(3) \qquad v^2/r = (k+1)N, \qquad vv_s = T.$$

The result is

$$(4) \qquad Nr_s = (n+1)T - r\mathfrak{N},$$

where

$$(4') \qquad n = 2/(k+1).$$

We may readily find various properties from this intrinsic equation, but in order to obtain a complete set it is necessary to have recourse to the equivalent equation in cartesian coordinates

$$(5) \qquad \begin{aligned}(\psi - y'\varphi)y''' = \{\psi_x + (\psi_y - \varphi_x)y' - \varphi_y y'^2\}y'' \\ - \left\{3 + \frac{(n-2)(\varphi + y'\psi)}{1+y'^2}\right\}y''^2.\end{aligned}$$

This obviously reduces to the familiar trajectory equation of §1 when $n = 2$, corresponding to $k = 0$. Brachistochrones correspond to $n = -2$, catenaries to $n = 1$, velocity curves to $n = 0$.

107. We now state the characteristic properties of a system of the above type for any value of n, that is, any value of k.

Characteristic Properties of the System S_k

Property 1.—For any given element (x, y, y') the foci of the osculating parabolas of the single infinity of curves determined by the given element lie on a circle passing through the given point.

Property 2.—At any point O the tangent of the angle which the focal circle makes with the given element is to the tangent of the angle which the given element makes with a certain direction fixed at O (the direction of the acting force) as 3 is to $n + 1$, that is, as $3k + 3$ is to $k + 3$.

Property 3. Through a given point there pass a single infinity of curves admitting hyperosculating circles of curvature; the centers of these circles lie on a conic passing through the given point in the direction of the force vector.

Property 4.—The normal at the given point O cuts the conic described in property 3, at a distance equal to $n + 1$, that is $(k + 3)/(k + 1)$, times the radius of curvature of the line of force passing through O.

Property 5.—This is of the same form as property V (§ 3) obtained in the discussion of trajectories, the number 3 being replaced by the number $n + 1$. In the notation of page 11

$$\frac{\partial}{\partial x}\frac{1}{AA'} + \frac{\partial}{\partial y}\frac{1}{BB'} + \frac{\omega\omega_{xy} - \omega_x\omega_y}{(n + 1)\omega^2} = 0.$$

. 108. The special case where n equals $- 1$, that is, the system S_{-3}, is exceptional and requires a separate discussion; but as we do not need the results, this case is omitted.

109. While the properties corresponding to different values of k are analogous, they are of course not identical. The first property is common to all the systems. But the second property involves the parameter k. Thus, while for trajectories the constant ratio that appears is 1 (bisection), it is $- 3$ for brachistochrones, 3/2 for catenaries, and 3 for velocity curves. Not only are the triply infinite systems S_k, corresponding to different values of k, distinct in any given field of force, but also no two

systems arising in two distinct fields can ever coincide. For example, if a certain system of ∞^3 curves arises as trajectories in one field, it cannot also arise as catenaries in either the same or another field.

110. If we combine all the systems S_k, in a given field (φ, ψ), we obtain a quadruply infinite system which we now proceed to study. The differential equation of the fourth order defining this system is readily obtained by eliminating k from the equation of S_k. It is more convenient to carry this out in terms of intrinsic quantities, using either the radius of curvature and its first and second derivatives with respect to the arc, quantities denoted by r, r_s, r_{ss}, or else the radius of curvature together with the radii of the first and second evolute, quantities which we denote by r, r_1, r_2. The two sets of quantities are equivalent, being connected by the relations $r_1 = rr_s$, $r_2 = r^2r_{ss} + rr_s^2$. The equation of the quadruply infinite system may then be put, using the notation of § 2, into the form

$$\begin{vmatrix} Nr_s + r\mathfrak{N} & T \\ Nr_{ss} + \left(2\mathfrak{N} - \dfrac{T}{r}\right) r_s & \mathfrak{T} + \dfrac{N}{r} \end{vmatrix} 0 = .$$

This may be written in either of the forms

$$r_{ss} = (\beta_1 + \beta_2 r^{-1}) r_s + \beta_3 r + \beta_4,$$

$$r_2 = r^{-1}r_1^2 + (\beta_1 r + \beta_2) r_1 + \beta_3 r^3 + \beta_4 r^2,$$

where the β's are functions of x, y, y'.

111. We notice first that r_2 is quadratic with respect to r_1. Hence for given values of x, y, y', r, that is for a given curvature element, the ∞^1 curves of the system have the property that the locus of the third center of curvature is a parabola with axis parallel to the fixed radius of curvature, that is, perpendicular to the initial direction y'.

112. An equivalent statement is this: If for each of the curves we construct the osculating conic (five-point contact), the locus

of the centers of these conics is a conic passing through a given point in the given direction. It is perhaps worth while to restate this, so far as it concerns the four special cases of physical interest, as follows: In any plane field of force select any fixed element of curvature; corresponding to the initial values of x, y, y' and r so given, construct the unique trajectory, unique brachistochrone, unique catenary, the unique velocity curve, and the respective centers of the osculating conics; the four centers so found and the given point (x, y) will lie on a conic passing through the latter point in the given direction y'. (Cf. the first footnote on page 98.)

113. Keeping the curvature element fixed and varying the parameter k, the value of r_s or, what is equivalent, of r_1, varies linearly. As above, let n denote the fraction $2/(k+1)$; then if values of n forming an arithmetic progression are selected, the corresponding values of r_1 also form an arithmetic progression. The successive differences in the values of r_1 corresponding to the case of trajectory, brachistochrone, catenary, and velocity curve are proportional to 4, -3, 1.

114. If in the system S_k we keep x, y, y' fixed and vary r, two limiting cases of interest arise. First, if r becomes infinite, then r_s is also infinite, and the limiting curve obtained is a straight line. In fact the ∞^2 straight lines of the plane form part of every system S_k.

On the other hand, if r approaches zero, then r_s approaches a definite limit

$$(n+1)T/N.$$

Remembering that the tangent of the angle of deviation is one third of r_s, we may state the result obtained as follows: In any system S_k if we take any lineal element and let r approach zero, the tangent of the corresponding angle of deviation is to the tangent of the angle which the force vector makes with the normal to the given element in the fixed ratio of $n+1$ to 3. The special values of this ratio for the four special systems of physical interest are respectively 1, $-1/3$, $2/3$, $1/3$. In the case of trajectories, it is noteworthy that the limiting position of the axis

of deviation coincides with the direction of the force acting at the given point.

§§ 115–116. Curves of Constant Pressure

115. We now consider a second simple generalization of the problem $P = 0$, defining trajectories. We consider, namely, curves corresponding to $P = c$, where c denotes any constant. The curves obtained may be termed curves of constant pressure: only along such a curve is a constrained motion of a particle possible such that the pressure against the curve remains constant.

For a given value of c a system of ∞^3 such curves is obtained, whose intrinsic equation, found by differentiating the relation

$$P \equiv v^2/r - N = c,$$

is

$$(c + N)r_s = 3T - \mathfrak{N}.$$

We see that this system for any value of c retains property I of the system of trajectories. Omitting the discussion of the higher properties of these triply infinite systems we consider the quadruply infinite system whose differential equation, found by eliminating c, may be written in either of the intrinsic forms

$$(\mathfrak{N}r^2 - 3Tr)r_{ss} = (2r\mathfrak{N} - T)r_s^2 + [\mathfrak{N}_1 r^2 + (\mathfrak{N}_2 - 3\mathfrak{T})r - 3N]r_s,$$

$$r(\mathfrak{N}r - 3T)r_2 = (3r\mathfrak{N} - 4T)r_1^2 + [\mathfrak{N}_1 r^2 + (\mathfrak{N}_2 - 3\mathfrak{T})r - 3N]rr_1.$$

This gives the totality of ∞^4 curves of constant pressure defined by a given field.

As regards special cases of interest, we note, in addition to $c = 0$, giving trajectories, the case $c = \infty$ which gives $r_s = 0$, defining circles; hence for any field of force the ∞^4 curves of constant pressure include the ∞^3 circles of the plane, which arise in fact as curves of infinite pressure.

116. The quadruply infinite system which here arises, as well as that obtained in the previous problem $P = kN$, comes under

the category represented by a differential equation of the type*

$$y^{\mathrm{IV}} = Ay'''^2 + By''' + C.$$

It therefore enjoys the property, previously stated in the other problem (§ 112), that the locus of the centers of the osculating conics corresponding to any element (x, y, y', y'') is a conic touching the element (x, y, y'). Of course, since the forms of A, B, C in the two problems are quite distinct, the systems are distinguished in their higher properties.

§§ 117–118. Tautochrones

117. Tautochrones are not included in either of the previous problems. They are not distinguished by any simple law of pressure.† The condition for a tautochrone is that the resulting constrained motion of a particle along the curve be harmonic, that is,

$$T = k(s - s_0), \tag{1}$$

where k is a constant (which is negative for actual and positive for virtual tautochrones) and $s - s_0$ denotes the arc reckoned from a fixed point of the curve, the center of the tautochronous motion. From this

$$T_{ss} = 0 \tag{2}$$

and hence, by expansion, *the general equation of the system of* ∞^3 *tautochrones in any field is*‡

$$Nr_s = \mathfrak{X}_1 r^2 + (\mathfrak{X}_2 + \mathfrak{N})r - T, \tag{3}$$

where the notation is that of § 2.

* This type (noteworthy in that it unifies many distinct mathematical and physical problems) first presented itself in the author's study of "Systems of extremals in the calculus of variations," *Bull. Amer. Math. Soc.*, vol. 13 (1907), p. 290: the extremals of any integral of the second order $\int f(x, y, y', y'')dx$ form a system of that type. In these lectures other physical problems leading to species included in this type are treated in §§ 110, 135, 137.

† It may be shown that during any tautochronous motion

$$P = k(s - s_0)^2/r - N.$$

‡ "Tautochrones and brachistochrones," *Bull. Amer. Math. Soc.*, vol. 15 (1909), pp. 475–483.

We see that r_s is a quadratic function of r, and not a linear function as in the case of trajectories and the other systems S_k. For a discussion of the geometric properties of tautochrones, we refer to the dissertation of H. W. Reddick.*

118. There is no field in which the tautochrones coincide with the trajectories, or with any of the systems S_k, in either the same or some other field, except for the case $k = -2$ corresponding to brachistochrones. The classical work of Huygens and J. Bernoulli showed that for a uniform field the system of tautochrones is identical with the system of brachistochrones. The author has shown that the only other field where such duplication occurs is that in which the force is central and varies directly as the distance. The only case of duplication in two distinct fields is as follows: The tautochrones of the field $\varphi = 0$, $\psi = y$ coincide with the brachistochrones of the field $\varphi = 0$, $\psi = y^{-3}$. The particular fields arising in this duplication problem are included in the interesting class of fields, involving eight parameters, characterized by the vanishing of the element function T_1. For such a field r_s, according to (3), becomes linear in r, and hence the ∞^2 straight lines of the plane are included in the system of tautochrones.†

118′. Each of the ∞^3 tautochrones in a given field has associated with it a certain time of oscillation, determined by the value of the constant k in (1). To each value of the period, that is, to each value of k, corresponds a certain family of ∞^2 tautochrones, whose differential equation, in implicit form, is

$$r(k - \mathfrak{T}) = N,$$

or, expanded,

$$(\psi - y'\varphi)y'' = k(1 + y'^2) - \{\varphi_x + (\varphi_y + \psi_x)y' + \psi_y y'^2\}.$$

We pass over the easy geometric interpretation; and note merely the special family, corresponding to the value $k = 0$, for which

* *Amer. Jour. of Math.*, vol. 33 (1911).

† The corresponding problem in space is treated in Reddick's paper and gives a class of fields involving twenty parameters.

the period is infinite. This separates the actual from the virtual tautochrones.

§ 119. Non-uniform Catenaries

119. It is a familiar fact that vertical parabolas appear in elementary dynamics in two distinct discussions; first, as trajectories of a cannon ball, and secondly as forms of equilibrium of a chain in which the mass (or load) of any element is proportional to the horizontal projection of that element. Here the force is ordinary gravity. The question arises whether any other fields of force give rise to a like duplication.

We first consider the following general problem of non-uniform catenaries. If a flexible string or chain, in which the mass of any element of length is proportional to some given function μ of x, y, y', is suspended in a positional field, the possible forms of equilibrium are defined by the equation

$$Nr_s = 2T - (1 + y'^2)N\bar{\mu}_{y'} - r\{\Re + (1 + y'^2)^{-\frac{1}{2}}N(\bar{\mu}_x + y'\bar{\mu}_y)\}.$$

This represents the ∞^3 non-uniform catenaries for a given field $\varphi(xy)$, $\psi(xy)$ and a given density law $\mu(x, y, y')$, where μ denotes $\log \mu$.

On the other hand, the trajectories in the given field are defined by the equation

$$Nr_s = 3T - r\Re.$$

Our problem then is to find those fields for which the two systems described coincide. *The result obtained is that the field must be central or parallel.* The detailed result is as follows:

In any central field of force the ∞^3 trajectories may be also obtained as catenaries by loading the chain so that its density is proportional to the perpendicular dropped from the center to the tangent line. In the more special case where the field is parallel, the density is proportional to the sine of the angle between the element of the curve and the force.

It is easy to obtain analogous comparisons between brachistochrones and catenaries. In this case the density must vary inversely as the cube of the perpendicular dropped from the center (or of the sine of the angle referred to above). For example, in the case of gravity the vertical cycloids which appear as brachistochrones may be obtained as catenaries by causing the load applied to any element to vary inversely as the cube of its horizontal projection.

All the results may be included in a generalization found by comparing the non-uniform catenaries with the systems denoted by S_k in § 103. The density must vary as the $(n-1)$th power of the perpendicular, where n is the number defined on page 93. The field is necessarily central or parallel.

CHAPTER V

MORE COMPLICATED TYPES OF FORCE

§§ 120–122. Motion in a Resisting Medium

120. We consider the motion of a particle moving in the plane under a positional field of force and influenced by a resisting medium, the resistance acting in the direction of the motion and varying as some function of the speed v.* The equations of motion will then be of the form

$$(1)\qquad \ddot{x} = \varphi(x, y) + \dot{x}f(v), \qquad \ddot{y} = \psi(x, y) + \dot{y}f(v),$$

where the resistance R is equal to

$$R = vf(v).$$

The differential equation of the trajectories is found to be

$$(2)\qquad \begin{aligned}(\psi - y'\varphi)y''' = \{\psi_x + y'(\psi_y - \varphi_x) - y'^2\varphi_y\} \\ - 3\varphi y''^2 - 2f\sqrt{\psi - y'\varphi}\, y''^{\frac{3}{2}},\end{aligned}$$

where the argument v of f is to be expressed in terms of x, y, y', y'' by means of

$$v^2 = \frac{(\psi - y'\varphi)(1 + y'^2)}{y''}.$$

Consider now the ∞^1 trajectories starting from a given element (x, y, y'). The focal locus, that is, the locus of the foci of the osculating parabolas, varies in shape with the function f, that is, with the law of resistance.

We know that, if there is no resistance, property I of § 3 holds, that is, the focal locus is a circle passing through the given point. Are there any resisting media for which this property is preserved? A simple discussion shows that there are, the appro-

priate media being those for which R is of the form $Av^2 + B$.

For such media, property II will not usually be fulfilled; in fact *the only medium preserving the properties I and II is that in which the resistance varies as the square of the speed.*

If we impose also property III, both A and B must vanish, that is, the resistance vanishes and the force is purely positional.

It is of interest to examine the case where the resistance varies as any power v^n of the speed. The differential equation of the trajectories is then of the form

$$y''' = ay'' + by''^2 + cy''^m,$$

where

$$m = \tfrac{1}{2}(4 - n).$$

The focal locus is a curve whose inverse with respect to the given point is

$$X = a_1 + b_1 Y + c_1 Y^{m-1}$$

This becomes a straight line (as in the case of no resistance), when m is 1 or 2, that is, when n is 2 or 0.

The curve is a conic when m is 3 or 0 or 3/2, that is, when n has one of the values -2 or 4 or 1. When $n = -2$ the conic is a parabola with its axis parallel to the given element. When $n = 4$ it is a hyperbola, asymptotic to the line of the given initial element. When $n = 1$ it is a parabola touching the initial line (not at the given point).

121. We now state briefly the corresponding results in ordinary space. No matter what the law of resistance is, property I (of the set of four properties for space given in § 11) is fulfilled; for the osculating planes necessarily pass through the force vector. The only laws for which property II is preserved are those included in

$$R = Av^2 + B.$$

If property III is also to be preserved, the resistance must vanish.

122. The results may be derived easily from the intrinsic equations

(3) $$v^2 = rN, \qquad vv_s = T + R,$$

obtained by taking components of the acting forces along the normal and tangent to the trajectory. The geometric equation, resulting from the elimination of v, is of the form*

$$(4) \qquad Nr_s = -r\mathfrak{R} + 3T + 2R.$$

This gives the relation between r_s (the rate of variation of r with respect to s) and r (the radius of curvature). The resistance R, which is given as a function of v, is here to be expressed in terms of r by means of the first of the relations (3). If property I, of plane trajectories, is to hold, r_s must be a linear integral function of r; this will be the case not only when R vanishes, but also, as stated above, when it is of the form $Av^2 + B$.

§§ 123–126. Particle on a Surface

123. The motion of a particle on any constraining surface

$$x = \varphi(u, v), \qquad y = \psi(u, v), \qquad z = \chi(u, v)$$

under any positional forces may be investigated most simply by means of the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial \dot{v}} = U, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = V'$$

where T is the kinetic energy

$$2T = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

and U, V are the components of the force given as functions of u, v.† The explicit equations of motion are of the form

$$\begin{aligned} \ddot{u} &= \Phi + A_0\dot{u}^2 + 2A_1\dot{u}\dot{v} + A_2\dot{v}^2, \\ \ddot{v} &= \Psi + B_0\dot{u}^2 + 2B_1\dot{u}\dot{v} + B_2\dot{v}^2; \end{aligned}$$

* From this we may obtain the following dynamical result: If a particle starts from rest, the initial radius of curvature of the trajectory is to the radius of curvature of the line of force passing through the initial point as $3T + 2R$ is to T. When R vanishes we have the simple result previously stated.

† See for example Whittaker, Analytical Dynamics, p. 390, and Hadamard, *Jour. de Math.* (5), vol. 3, p. 331.

where Φ, Ψ define the force and the A's and B's are functions of u, v depending only on the given surface.

124. We observe that here $\ddot{u}$, $\ddot{v}$ depend not only on the position u, v but also upon the velocity $\dot{u}$, $\dot{v}$. Hence the motion in the uv-plane corresponding to the actual motion on the surface is not usually generated by any positional force in that plane. The only exception arises when the A's and the B's vanish identically: this is the case only if the given surface is developable, and if its representation on the uv-plane differs from its development on the plane by at most an affine transformation.

Another problem including this as a special case is to determine when the motion in the uv-plane can be regarded as due to a positional force together with a resistance acting in the direction of the motion. The condition for this is

$$\frac{A_0\dot{u}^2 + 2A_1\dot{u}\dot{v} + A_2\dot{v}^2}{B_0\dot{u}^2 + 2B_1\dot{u}\dot{v} + B_2\dot{v}^2} = \frac{\dot{u}}{\dot{v}}.$$

Expanding, we find four conditions on the six functions A, B, which turn out to be precisely the conditions that the geodesics of the surface shall be pictured by straight lines, a result which may be proved directly. Hence the only case in which the motion on the surface is pictured in the uv-plane by a motion due to a positional force together with a resistance depending on the velocity components and acting in the direction of the motion, is that in which the surface has constant curvature and the representation is geodesic.

125. We proceed with the general equations of motion. If we eliminate the time, we obtain the differential equation of the third order defining the ∞^3 trajectories in the form

$$\begin{aligned}(\Psi - v'\Phi)v''' = \{\delta_0 + \delta_1 v' + \delta_2 v'^2 + \delta_3 v'^3 + \delta_4 v'^4 + \delta_5 v'^5\}& \\ + \{\epsilon_0 + \epsilon_1 v' + \epsilon_2 v'^2 + \epsilon_3 v'^3\}v'' - 3\Phi v''^2&,\end{aligned}$$

where the coefficients are functions of u, v. We confine ourselves to the observation that the picture curves in the uv-

plane come under the type

$$v''' = F_0 + F_1 v'' + F_2 v''^2,$$

where the coefficients are lineal-element functions: the focal locus is thus not a circle, but a special quartic. Hence if we consider the ∞^1 trajectories on the surface obtained by starting a particle at a given point in a given direction with different speeds, the picture curves in the uv-plane have osculating parabolas at the common point whose foci lie on a special quartic curve.

126. What is the simplest property of the actual trajectories described on the surface? What is, in particular, the locus of the osculating spheres of the ∞^1 trajectories considered?

To answer this we take our surface not in parametric form, but in the explicit form

$$z = f(x, y).$$

We may take the given point as origin, the tangent plane as the xy-plane, and the fixed initial direction as that of the axis of x. We find, by differentiating the equation of the surface and making use of $y' = 0$, $z' = 0$, that

$$z'' = a, \qquad z''' = b + cy'',$$

where a, b, c are constants, equal respectively to the values of the partial derivatives f_{xx}, f_{xxx}, $4f_{xy}$ at the origin. Again, from the general equation of the trajectories, we have a relation of the form

$$y''' = \alpha + \beta y'' + \gamma y''^2.$$

The center of the osculating sphere of the trajectory is then

$$X = 0,$$

$$Y = \frac{z'''}{y''z''' - z''y'''} = \frac{b + cy''}{y''(b + cy'') - a(\alpha + \beta y'' + \gamma y''^2)},$$

$$Z = \frac{-y'''}{y''z''' - z''y'''} = \frac{-(\alpha + \beta y'' + \gamma y''^2)}{y''(b + cy'') - a(\alpha + \beta y'' + \gamma y''^2)}.$$

Here y'' enters as parameter, varying from curve to curve: eliminating it, we find the locus, lying in the plane $X=0$, to be

$$\alpha Y^2 + \beta Y(1-aZ) + \gamma(1-aZ)^2 + Z\{bY + c(1-aZ)\} = 0.$$

Hence for any positional force on any surface, the ∞^1 trajectories starting from a given lineal element of the surface have osculating spheres, at the common point, whose centers lie on a (general) conic in the plane normal to the element.

This conic passes through the center of curvature of the normal section of the surface determined by the given element. If the element is in one of the principal directions of the surface, the conic touches the normal to the surface.

§§ 127–130. The General Field in Space of n-Dimensions

127. Any dynamical system with n degrees of freedom may be represented by a particle in space of n dimensions. For example, an arbitrary rigid body in ordinary space is represented by a particle in six-dimensional space, and the astronomical problem of three bodies in the most general case leads to a representative particle in space of nine dimensions.

For conservative forces, or natural families, the general discussion for any dimensionality has already been given (§ 69). We shall not attempt a complete discussion for arbitrary positional forces (corresponding to that given in Chapter I for two and three dimensions). The equations of motion for an arbitrary field are

$$\ddot{x}_1 = \varphi_1(x_1, \cdots, x_n), \qquad \cdots, \qquad \ddot{x}_n = \varphi_n(x_1, \cdots, x_n).$$

We confine ourselves to the simplest questions. If the initial position and initial direction are kept fixed, and only the initial speed v is varied, what are the properties of the ∞^1 trajectories obtained? The simplest geometric result is that r_s (the rate of variation of the radius of curvature with respect to the arc length) varies as a linear function of r. The locus of the centers of the osculating spheres is a straight line, just as in the case where n is three.

128. A general curve in n-space has at each point an osculating plane, an osculating 3-flat, and so on up to an osculating $(n-1)$-flat. It is obvious that our ∞^1 trajectories have the same osculating plane since this is determined by the given initial direction and the direction of the force. It can be shown that the osculating 3-flat is also fixed; the 4-flat varies, generating a pencil; the 5-flat varies, generating a quadratic system; and so on, with more complicated variations.

129. Consider next the connection between the various curvatures and the speed.

In the plane ($n = 2$) there is only one curvature γ_1, and this varies inversely as the square of v.

In space ($n = 3$) the first curvature γ_1 varies as above, and the second curvature or torsion γ_2 remains fixed.

If $n = 4$, we have three curvatures. The laws for γ_1 and γ_2 are as above, while

$$\gamma_3 = c_1 + c_2 v^{-2},$$

where c_1, c_2 are constants (depending of course on the given initial lineal element).

If $n = 5$, we have $\gamma_1 = av^{-2}$, $\gamma_2 = b$ (these forms are valid for any dimensions) and

$$\gamma_3 = \sqrt{c_1 + c_2 v^{-2} + c_3 v^{-4}}, \qquad \gamma_4 = \frac{d_1 + d_2 v^2 + d_3 v^4}{d_4 + d_5 v^2 + d_6 v^4}.$$

If $n = 6$, γ_3 remains the same, the numerator in γ_4 is replaced by the square root of a polynomial involving v^8, and γ_5 is given by a rational formula.

It is easy to write down the general formulas for the $n - 1$ curvatures in n space. All except the first, second, and the last are irrational. These results are to be regarded as generalizations of the elementary fact (included in the formula for centrifugal force v^2/r), that the ordinary curvature varies as v^{-2}.

130. By eliminating v from any two of the formulas, we can obtain purely geometric results. For example, in space of four dimensions, $\gamma_3 = A + B\gamma_1$, where A and B depend only on the

common initial element. But in higher spaces

$$\gamma_3 = \sqrt{A + B\gamma_1 + C\gamma_1^2}.$$

This is the form required in particular in the application to the problem of three bodies, since the representative space has nine dimensions.

§§ 131–132. Interacting Particles in the Plane and in Space

131. We consider the motion of $n + 1$ particles, denoted by $M, M_1, \cdots, M_n$, moving in the plane under the action of any forces depending on the position of the particles. The differential equations of motion are then of the form

$$\begin{aligned}\ddot{x} &= \varphi(x, y, x_1, y_1, \cdots, x_n, y_n),\\ \ddot{y} &= \psi(x, y, x_1, y_1, \cdots, x_n, y_n),\\ \ddot{x}_1 &= \varphi_1(x, y, x_1, y_1, \cdots, x_n, y_n),\\ \ddot{y}_1 &= \psi_1(x, y, x_1, y_1, \cdots, x_n, y_n),\end{aligned}$$

and so on, where the masses—which cannot be assumed to be unity as in the case of a single particle—are absorbed with the forces in the right hand terms. From these equations the following properties may be deduced.

(1) Given the phases of $M_1, \cdots$, and the position and the direction of M, a set of ∞^1 trajectories of M is determined (one for each value of the speed). The foci of the osculating parabolas lie on a special quartic curve whose inverse with respect to the given point is a parabola tangent to the given initial line (the point of contact, however, is usually not the given point).

(2) If the speed of one of the remaining particles, say M_1, is varied, all the other initial conditions being unaltered, the parabolic locus just obtained varies. Its point of contact with the initial line remains fixed and all the ∞^1 parabolas, one for each value of the speed, are homothetic with respect to the point of tangency.

(3) The normal constructed at the common point of tangency cuts the parabola again at a distance d which varies in such a way that the square root of d can be expressed as a linear combination of the square roots of the radii of curvature of the corresponding trajectories described by the particles $M_1, \cdots, M_n$.

(4) If we preserve the phases of the particles $M_1, \cdots, M_n$, then, for each initial direction y' of M, we obtain, by (1), a certain parabolic locus. Consider the relation between the axis of this parabola and the initial direction. It is found that the initial direction y' always bisects the angle between the direction of the force acting at the given point and the direction of the axis of the parabola.

(5) Furthermore, the point where the parabola touches the initial line describes, when y' varies, a quartic curve whose inverse with respect to the given point is a conic passing through that point in the direction of the force.

It is to be observed that the statement (3) about the variation of d simplifies considerably in the case of *two* particles (that is, $n = 1$). In that case d varies directly as the radius of curvature of the trajectory described by M_1.

132. A few corresponding results for the case of any number of particles moving in space are as follows: If the speed of M is the sole arbitrary parameter, the ∞^1 trajectories of M have the same osculating plane; the torsion varies according to a linear integral function of the square root of the curvature; the locus of the centers of the osculating spheres is a cubic curve of special type.

If we assign the phases of all the particles except M_1 and assign the position and direction of M_1, then the speed of M_1, or, in consequence, the curvature of the trajectory described by M_1, is the only arbitrary parameter. There will then be ∞^1 corresponding trajectories described by M. These will of course start from the same point in the same direction with a common osculating plane and a common curvature, that is, they all have contact of the second order. The torsion varies and so does the

center of the osculating sphere. The simultaneous variation is controlled by the law that the distance from the center of the osculating sphere to the fixed center of curvature varies as a linear integral function of the radius of torsion. An equivalent statement is that the rate of variation of the radius of curvature per unit of the arc is expressed by a linear integral function of the torsion.

All these results apply in particular to the three-body problem. The present application is more concrete than that indicated in § 130, since no higher space is here introduced.*

§§ 133–141. Forces Depending on the Time. Trajectories and Space-Time Curves

133. Hitherto the force has been assumed to be independent of the time; now we consider the generalization where the force depends in any way upon the time as well as the position. Take the case of a particle moving in the plane; the equations of motion are then of the form

$$(1) \qquad \ddot{x} = \varphi(x, y, t), \qquad \ddot{y} = \psi(x, y, t).$$

From these, by differentiation and elimination, we may derive

$$(2) \qquad y''' = Py'' + Qy''^2 + Ry''^3,$$

where the coefficients are functions of x, y, y', t, namely,

$$P = \frac{\psi_x + y'\psi_y - y'(\varphi_x + y'\varphi_y)}{\psi - y'\varphi},$$

$$Q = \frac{-3\varphi}{\psi - y'\varphi}, \qquad R = \frac{\psi_t - y'\varphi_t}{(\psi - y'\varphi)^2}.$$

If we are given the initial time, position and direction, that is, the initial values of t, x, y, y', there will be a certain set of ∞^1

* Since the forces in the three-body problem are conservative, we may decompose the motions into natural families, and interpret each family in a flat space of eight dimensions. The circles of curvature at a given point will meet again; eight of them will be hyperosculating, and these will be mutually orthogonal. Cf. § 70.

trajectories, one for each value of the initial speed. The following properties are obtained:

(1) We find that the focal locus (that is, the locus of the foci of the ∞^1 osculating parabolas) is a quartic curve whose inverse with respect to the given point is a parabola which is tangent to the given direction line (the point of contact is not usually at the given point).

(2) As y' varies (x, y, t being held fixed) this point of contact describes a cubic curve whose inverse is a conic passing through the given point in the direction of the force.

(3) The initial direction of y' bisects the angle between the direction of the force and the direction of the axis of the parabola described in (1).

134. The total system of trajectories, for all initial conditions, consists of ∞^4 curves. Only in the case where the force does not depend upon the time does the system consist of ∞^3 trajectories. In the properties stated above, the initial time is kept fixed. In a certain sense then the results are not purely geometric: they would not appear in a photograph of the complete system of trajectories. This system will be represented by a certain differential equation of the fourth order; but it is not possible to carry out the requisite eliminations in explicit form, and hence the derivation of purely geometric properties involves essentially new difficulties. A complete characterization is however obtained, by projection from space curves, in §§ 136, 140.

135. There is an interesting special case in which the elimination can be carried out: namely, the problem of the motion of a *particle of variable mass* in a positional field of force. The time then appears only through the mass, so the equations of motion are of the form

$$(3) \qquad f(t)\ddot{x} = \varphi(x, y), \qquad f(t)\ddot{y} = \psi(x, y).$$

As the result of the elimination is complicated, we shall here consider only the case where the function $f(t)$, representing the mass, is of one of the special types t^4, t^2, e^t, $(\log t)^2$. The equa-

tion of the fourth order representing the trajectories is then found to be of the form

$$(4) \qquad y^{\mathrm{IV}} = Ay'''^2 + By''' + C,$$

where A, B, C involve only x, y, y', y''.

We see that the fourth derivative is a quadratic function of the third derivative. This category of equations of the fourth order arises in a number of different connections, in particular in the inverse problem of the calculus of variations, as stated in § 116. The characteristic geometric property may in the present case be stated as follows:

If the particle, whose mass varies according to one of the four laws stated, is projected into a field of force from a fixed initial position in a fixed direction at different times, with the initial speed for each time so adjusted as to cause the initial curvature of the trajectory to have a fixed value, and if for each of the ∞^1 trajectories thus obtained we construct the osculating conic (having five-point contact), the locus of the centers of these conics is a conic passing through the given conic in the given direction.

Of course not every system of ∞^4 curves having this property can be regarded as a trajectory system corresponding to equations of motion of the form considered. We do not, however, attempt a complete characterization.

136. *Space-time Curves.*—When we integrate the equations of motion, either in the special case where the forces depend only on the position

$$(1') \qquad \ddot{x} = \varphi(x, y), \qquad \ddot{y} = \psi(x, y),$$

or in the general case where the force depends also on the time

$$(1) \qquad \ddot{x} = \varphi(x, y, t), \qquad \ddot{y} = \psi(x, y, t),$$

we obtain x and y expressed as functions of t and four constants of integration. If we represent t by an ordinate perpendicular to the xy-plane, thus considering x, y, t as rectangular coordinates

in space, we obtain a certain system of ∞^4 curves in that space which we designate as *space-time curves.**

If we project these curves orthogonally on the xy-plane, we obtain the trajectories. In the general case (1) there will be ∞^4 of these trajectories; but in the special case where the force is positional, only ∞^3 trajectories arise, since the system of space-time curves, whose number is still ∞^4, now admits the group of translations along the t-axis.

If we project the space-time curves orthogonally on the xt-plane and on the yt-plane, we obtain in each case a system of ∞^4 plane curves.

What are the properties of the system of ∞^4 space-time curves? The following two properties are characteristic:

(1). The osculating planes of the ∞^2 space-time curves through a given point go through a fixed line parallel to the xy-plane. (This line is parallel to the direction of the force acting at the projected point in the xy-plane.)

(2). If the ∞^2 space-time curves through the given point are orthogonally projected on any plane perpendicular to the xy-plane, the ∞^2 plane curves obtained are such that those which have the same tangent also have the same curvature.

Another complete characterization may be given as follows:

(3). If the ∞^2 space-time curves through a given point are orthogonally projected on either the xt-plane or the yt-plane, the ∞^2 plane curves obtained have their centers of curvature located on a special cubic of the form $t^3=a(x^2+t^2)$ or $t^3=b(y^2+t^2)$. A corresponding cubic locus will then necessarily arise by projection on any plane perpendicular to the xy-plane.

* It may be remarked that if, in problem (1), the force is multiplied by a constant c (or, what is equivalent, the mass of the particle is multiplied by $1/c$), a distinct system of ∞^4 space-time curves will be obtained. The totality of ∞^5 space curves, thus related to the ∞^1 plane problems

$$\ddot{x} = c\varphi(x, y, t), \qquad \ddot{y} = c\psi(x, y, t),$$

may be generated as trajectories in a three-dimensional positional field of force. The ∞^5 curves have the four characteristic properties of a space system (§ 11) and the further peculiarity that the direction of the force is parallel to the xy-plane.

137. Consider the ∞^4 curves in say the xt-plane. These are the curves representing graphically the relation between the abscissa x and the time t. By eliminating y from the set (1), we obtain a relation of the form

$$x^{\text{IV}} = A\dddot{x}^2 + B\dddot{x} + C,$$

where A, B, C involve only x, $\dot{x}$, $\ddot{x}$ and the independent variable t. The fourth derivative is thus always quadratic with respect to the third derivative. Hence, by § 116, we have this result:

In the xt-plane (or, more generally, in any plane perpendicular to the plane xy in which the motion actually takes place), the ∞^1 curves having any element of curvature in common are such that the locus of the centers C' of their osculating conics (constructed at the common point) is a conic passing through the common point in the direction of the common tangent.

As indicated above, the ∞^4 curves in the xy-plane, that is, the trajectories, do not usually enjoy this simple property. Even in the case where the time enters only through the mass, the locus of the centers of the osculating conics may be of any degree of complication. Its shape depends on the law of variation of the mass. Only for the special laws stated at the bottom of page 112, together with certain combinations of them, is the equation of the trajectories of the quadratic type.

138. It is possible to obtain additional general properties of the xt-system, describing how the locus conic, corresponding to a curvature element, changes when the element changes. For the coefficients A, B, C determining the position of the conic have the following forms: A does not involve $\dot{x}$, B is linear and integral in $\dot{x}$, C is quadratic and integral in $\dot{x}$. Hence these results:

If the curvature element is varied, at the given point O, in such a way that the second derivative $\ddot{x}$ is constant, so that only $\dot{x}$ varies, the center C'' of the corresponding locus conic describes a new conic.

At the same time a certain two-to-one correspondence arises between the initial direction of the element and the direction of the line joining O to the center C''.

139. A clearer picture is perhaps obtained by changing the notation to correspond with the usual x, y, z notation for rectangular coordinates in space. It is then desirable to lay off the time on the x-axis, since this is the independent variable. The actual motion then takes place in the yz-plane, and the differential equations of motion are

$$\frac{d^2y}{dx^2} = \varphi(x, y, z), \qquad \frac{d^2z}{dx^2} = \psi(x, y, z).$$

The curves in space x, y, z are then the space-time curves. Their projections on the yz-plane are the trajectories (whose explicit properties have not been derived). Their projections on the xy-plane (or on the xz-plane, or on any plane parallel to the z-axis) are curves whose properties have just been stated (§§ 137, 138). The differential equation in the xy-plane is

$$y^{\text{IV}} = \alpha y'''^2 + (\beta_1 + \beta_2 y')y''' + (\gamma_1 + \gamma_2 y' + \gamma_3 y'^2),$$

where the coefficients involve only x, y, and y'.

140. We have not attempted a complete direct characterization of the systems of curves arising in any one of the coordinate planes. Such a characterization has however been given (§ 136) for the system of ∞^4 space-time curves. Indirectly this really solves all the problems. A system of curves in the plane can be regarded as trajectories of a force depending on time and position if and only if the curves can be obtained by orthogonal projection from some system of ∞^4 curves in space having the properties (1) and (2) of § 136. If, furthermore, the space system is invariant under translation perpendicular to the given plane, the plane system, then consisting of only ∞^3 curves, belongs to a positional field.

141. For any force depending on time and position

$$\ddot{x} = \varphi(x, y, t), \qquad \ddot{y} = \psi(x, y, t),$$

the number of space-time curves is always ∞^4. When we project

these on the xy-plane, to obtain the trajectories, the number is usually ∞^4. The number reduces to ∞^3 if the force is positional but does not vanish; in the latter case the trajectories are merely the ∞^2 straight lines.

In the xt-plane the usual number of curves is ∞^4. The only exception arises when the function φ is free from the variable y. In this case the xt-curves all satisfy the equation of second order $\ddot{x} = \varphi(x, t)$ and therefore their number is only ∞^2. Similar statements hold of course for the yt-plane.

Consider, as a single example, gravity, taken as uniform and acting in the vertical xy-plane. The equations of motion are

$$\ddot{x} = 0, \qquad \ddot{y} = g.$$

The xyt-curves are

$$x = at + b, \qquad y = \tfrac{1}{2}gt^2 + ct + d,$$

a certain family of ∞^4 parabolas in space. The xt-curves are ∞^2 straight lines. The yt-curves are ∞^2 parabolas. The xy-curves (that is, the trajectories) are ∞^3 parabolas

$$y = \alpha x^2 + \beta x^2 + \gamma.$$

It is to be observed that if the gravity constant g is changed, the new problem, while giving the same trajectories, gives a distinct family of xyt-curves. If g takes all possible values, the totality of space-time curves obtained is formed of ∞^5 parabolas (namely, those whose axes are parallel to the t-axis). These curves, in accordance with the general statement made in the footnote on page 114, are the trajectories of a positional field in space, the generating force being constant and acting in the t-direction.

All the results can be extended so as to apply to the four-dimensional space-time curves depicting motion in ordinary space.